Average Kullback-Leibler Divergence for Random Finite Sets

Giorgio Battistelli, Luigi Chisci
and Claudio Fantacci
Dipartimento di Ingegneria dell’Informazione
Università di Firenze
Firenze 50139, Italy
e-mail: giorgio.battistelli@unifi.it
e-mail: luigi.chisci@unifi.it
e-mail: claudio.fantacci@unifi.it

Alfonso Farina
Selex ES
(retired at the end of 2014)
Rome 00131, Italy
e-mail: alfonso.farina@outlook.it

Ba-Ngu Vo
Department of Electrical and Computer Engineering
Curtin University
Bentley, WA 6102, Australia
e-mail: ba-ngu.vo@curtin.edu.au

Abstract—The paper deals with the fusion of multiobject information over a network of heterogeneous and geographically dispersed nodes with sensing, communication and processing capabilities. To exploit the benefits of sensor networks for multiobject estimation problems, like e.g. multitarget tracking and multirobot SLAM (Simultaneous Localization and Mapping), a key issue to be addressed is how to consistently fuse (average) locally updated multiobject densities. In this paper we discuss the generalization of Kullback-Leibler average, originally conceived for single-object densities (i.e. probability density functions) to (both unlabeled and labeled) multiobject densities. Then, with a view to develop scalable and reliable distributed multiobject estimation algorithms, we review approaches to iteratively compute, in each node of the network, the collective multiobject average via scalable and neighborwise computations.

Index Terms—multiobject estimation; sensor networks; distributed fusion; random finite sets; consensus; multitarget tracking.

I. INTRODUCTION

The recent breakthrough in wireless sensor technology has opened up the possibility to develop efficient surveillance/monitoring systems consisting of radio interconnections of a multitude of low cost and low energy consumption devices with sensing, communication and processing capabilities. To best exploit this emerging technology, it is however fundamental to redesign multiobject estimation algorithms taking into account the following issues: 1) the single node has limited sensing and computational capabilities and must also limit data transmission, which is primarily responsible for energy consumption; 2) processing must be carried out in a distributed fashion with no coordination of a central unit and in a scalable way with respect to the network size; 3) each node is unaware of the correlations existing between its own information and the information received from other nodes.

In recent years, consensus has emerged as a powerful tool for distributed computation over networks [1], [2] and has been widely used in distributed parameter/state estimation algorithms [3]–[10]. In particular, a consensus algorithm can be used for distributed averaging over a network; each node (agent) aims to compute the collective average of a given quantity by iterative regional averages, where the terms collective and regional mean over all network nodes and, respectively, over neighboring nodes only. Central to consensus for distributed state estimation is the notion of Kullback-Leibler average for probability density functions [10]. An important functionality of a surveillance/monitoring sensor network is multiobject estimation (i.e. the joint detection and state estimation of multiple objects of interest in a given area under surveillance/monitoring) for which consensus on multiobject information among the different nodes of the network becomes a key issue.

In this paper, we provide an overview of Kullback-Leibler average for multiobject probability density functions. In particular, we review the generalization of the Kullback-Leibler average to multiobject probability densities proposed in [11], and how this concept is exploited to develop an effective distributed multiobject estimation algorithm based on the Gaussian Mixture Cardinalized Probability Hypothesis Density (GM-CPHD) filter [12], [13]. We also review the recent application of Kullback-Leibler fusion for multiobject densities developed in [11] to the recently introduced labeled random finite set models [14]–[16], which have lead to more accurate distributed multitarget tracking algorithms proposed in [17].

The rest of the paper is organized as follows. Section II introduces the necessary background for modelling multiobjects as random finite sets and reviews different (both unlabeled and labeled) multiobject representations. Section III recalls from [11] the notion of Kullback-Leibler average (fusion) of multiobject distributions and, in particular, provides new analytical expressions for the Kullback-Leibler fusion of certain labeled multiobject densities. Section IV shows how consensus can be exploited in order to fuse, in a scalable fashion, multiobject densities over a peer-to-peer (coordination-free) network. Section V outlines the structure of a general distributed multiobject estimation algorithm based on Kullback-Leibler fusion and consensus. Section VI shows a case-study concerning the application of Kullback-Leibler fusion and consensus to multitarget tracking. Finally section
VII provides concluding remarks and perspectives for future work.

II. MULTIOBJECT REPRESENTATION

Notation

Throughout the paper, the following notation will be adopted. \( \mathbb{R}, \mathbb{R}_+, \mathbb{N} \) denote the sets of real, nonnegative real and, respectively, positive integer numbers. \( E[\cdot] \) denotes the expectation operator. \( |X| \) denotes the cardinality (number of elements) of the finite set \( X \). Given two real-valued functions \( f \) and \( g \) defined over the same domain, their inner product is defined as \( <f,g> \triangleq \int f \cdot g \) where the integral (possibly a set integral to be defined later) is extended to the whole domain. Then, the operators \( \oplus \) and \( \odot \) are defined as follows:

\[
f \oplus g \triangleq f g < f, g >^{-1}
\]

\[
\omega \odot f \triangleq f^\omega < f^\omega, 1 >^{-1}
\]

for any \( \omega > 0 \).

Multiobject estimation

Multiobject estimation aims to jointly detect an unknown number of objects of interest in a given area of competence and to estimate their states. This has relevant applications, for instance, in multitarget tracking [19]–[23], SLAM (Simultaneous Localization and Mapping) [24], and multisource estimation [25] wherein the objects of interest are respectively targets moving in the surveillance area, relevant elements (landmarks) of the environment surrounding a navigating robot or vehicle, sources diffusing heat or pollutants in a monitored area. Since the multioject to be estimated is characterised by a twofold randomness in both the number of objects and in their states, a natural approach that will be pursued in this paper is to represent it in terms of a Random Finite Set (RFS).

Random finite sets

An RFS \( X \subset \mathcal{X} \) is a random variable taking values in \( \mathcal{F}(\mathcal{X}) \), the collection of all finite subsets of the single-object state space \( \mathcal{X} \). While \( \mathcal{F}(\mathcal{X}) \) does not inherit the usual Euclidean notion of probability density from \( \mathcal{X} \), a measure-theoretic notion of probability density on \( \mathcal{F}(\mathcal{X}) \) is available [26]. However, we adopt the Finite Set Statistic (FISST) notion of density since it is convenient and by-passes measure theoretic constructs [27], [28]. Hereafter, the basic concepts of FISST needed for the subsequent developments will be briefly reviewed.

An RFS \( X \) is completely characterized by its multiobject density. Multiobject densities of RFSs are defined with respect to the reference measure \( \mu \) given by

\[
\mu(T) = \sum_{i=0}^{\infty} \frac{1}{i!K^i} \int_{\mathcal{X}} 1_T(\{x_1, \ldots, x_i\}) d(x_1, \ldots, x_i)
\]

for any (measurable) subset \( T \) of \( \mathcal{F}(\mathcal{X}) \). The measure \( \mu \) is analogous to the Lebesgue measure on \( \mathcal{X} \) (indeed it is the unnormalized distribution of a Poisson RFS with unit intensity \( u = 1/K \) when the state space \( \mathcal{X} \) is bounded). Moreover, it was shown in [26] that for this choice of reference measure, the integral of a function \( \pi : \mathcal{F}(\mathcal{X}) \rightarrow \mathbb{R} \), given by

\[
\int \pi(X)\mu(dX) = \sum_{i=0}^{\infty} \frac{1}{i!K^i} \int_{\mathcal{X}} \pi(\{x_1, \ldots, x_i\}) d(x_1, \ldots, x_i),
\]

is equivalent to Mahler’s set integral [28], which is defined for a generic function \( g(\cdot) \) as follows:

\[
\int_S g(\delta X) \Delta \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} g(\{x_1, \ldots, x_n\}) d(x_1, \ldots, x_n).
\]

In particular,

\[
\beta(S) \triangleq \text{Prob}(X \subset S) = \int_S f(X) \delta X
\]

gives the probability that the RFS \( X \) is included in the subset \( S \) of \( \mathcal{X} \), and \( f(\cdot) \) is called the FISST density. Note that while the FISST density \( f(\cdot) \) is not a probability density, it is equivalent to the multiobject probability \( \pi(\cdot) \) as shown in [26]. This result is key to the generalization of concepts involving probability density to multiobject FISST densities. Hence, in this work, we use the FISST density as a probability density.

The first-order moment of the multiobject density, better known as Probability Hypothesis Density (PHD) or intensity function, has been found to be a very successful characterization [28]. In order to define the PHD function, let us introduce the number of elements of the RFS \( X \) within \( S \subset \mathcal{X} \) which is clearly given by

\[
n(S) = \int_S \sum_{\xi \in X} \delta_\xi(x) dx
\]

where \( \delta_\xi(\cdot) \) is the Dirac delta centered at \( \xi \). The PHD function is defined such that the expected number of elements of \( X \) in \( S \) is obtained by

\[
E[n(S)] = \int_S d(x) dx.
\]

Without loss of generality, the PHD function can be expressed as

\[
d(x) = \frac{E[n]}{\pi} s(x)
\]

where

\[
\pi = E[n] = E[n(\mathcal{X})] = \sum_{n=0}^{\infty} np(n)
\]

\[
s(x) = d(x)/\pi.
\]

are respectively the expected number of objects and a single-object PDF, and \( p(n) \) denotes the cardinality PMF (Probability Mass Function) i.e. the probability that the RFS \( X \) have \( n \) elements.
Poisson and iid cluster RFSs

Hereafter, two commonly used unlabeled representations of multiobjects, i.e. Poisson and iid cluster RFSs (processes) will be reviewed. A Poisson RFS is uniquely characterised by its intensity function \( d(\cdot) \) as follows.

**Definition 1** - Given a function \( d(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+ \), an RFS on \( \mathcal{F}(\mathbb{X}) \) with multiobject density

\[
f(X) = e^{-\int d(x)dx} \prod_{x \in X} d(x) = e^{-\pi \|X\|} \prod_{x \in X} s(x) \tag{8}
\]

is called Poisson process with PHD function \( d(\cdot) \), or equivalently expected number of objects \( \pi = \int d(x)dx \) and location PDF \( s(\cdot) = d(\cdot)/\pi \).

An iid (independent identically distributed) cluster RFS is completely characterised by its intensity function \( d(\cdot) \) and cardinality PMF \( p(n) \).

**Definition 2** - Given a PMF \( p(n) \) on the nonnegative integers and a PDF \( s(\cdot) \) on \( \mathbb{X} \), an RFS on \( \mathcal{F}(\mathcal{X}) \) with multiobject density

\[
f(X) = |X|! \cdot p(|X|) \prod_{x \in X} s(x) \tag{9}
\]

is called iid cluster process with cardinality PMF \( p(\cdot) \) and location PDF \( s(\cdot) \), or equivalently PHD function \( d(\cdot) = \pi s(\cdot) \) where \( \pi = \sum_{n=0}^{\infty} np(n) \).

Comparing (8) and (9), it is clear that a Poisson RFS is nothing but a special case of iid cluster RFS wherein the number of objects is restricted to be Poisson-distributed with parameter \( \pi \), i.e. \( p(n) = e^{-\pi} \pi^n / n! \). Assuming that the multitarget RFSs are Poisson or iid cluster processes is at the basis of the PHD [29] or, respectively, Cardinalized PHD (CPHD) [12] filtering approaches to multitarget estimation. For this reason, Poisson and iid cluster RFSs will be also referred to in the sequel as PHD and, respectively, CPHD representations of multiobjects.

Labeled random finite sets

In certain applications of multitarget estimation (e.g. multitarget tracking) the aim is not only to estimate the number and the states of the objects, but also to keep track of their trajectories over time. To this end, the notion of label is introduced in the RFS framework [15]–[16] so that each object can be uniquely identified and its track be reconstructed. Let \( \mathcal{L} = \{ \ell_i : i \in \mathbb{N} \} \) be the discrete label set. To incorporate object identity, a label \( \ell \in \mathcal{L} \) is appended to the state \( x \in \mathbb{X} \) of each object and a multitarget is accordingly regarded as an RFS on the labeled state space \( \mathbb{X} \times \mathcal{L} \), i.e. a labeled state \( x = (x, \ell) \in \mathbb{X} \times \mathcal{L} \) is assigned to each object. Let \( \mathcal{L} : \mathbb{X} \times \mathcal{L} \rightarrow \mathcal{L} \) be the projector of the labeled state space into the label set so that \( \mathcal{L}((x, \ell)) = \ell \). Then, to avoid situations in which multiple objects have the same label, the following definition of labeled RFS is introduced.

**Definition 3** - A labeled RFS \( \mathcal{X} \) with state space \( \mathbb{X} \) and label set \( \mathcal{L} \) is an RFS on \( \mathbb{X} \times \mathcal{L} \) such that any realization satisfies

\[
|X| = |\mathcal{L}(X)|. \tag{10}
\]

Notice that the condition (10) imposes that all elements of \( X \) have distinct labels. The set integral (3) is extended to any function \( g : \mathcal{F}(\mathbb{X} \times \mathbb{L}) \rightarrow \mathbb{R} \) defined on a labeled RFS, as follows:

\[
\int g(X) \delta X \triangleq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\ell_1, \ldots, \ell_n \in \mathcal{L}} \int g(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\}) d(x_1, \ldots, x_n). \tag{11}
\]

Hereinafter labeled states, spaces, multiobject densities will be denoted by bold symbols, e.g. \( x, \mathcal{X}, f(\cdot) \) instead of their unlabelled counterparts \( x, X, f(\cdot) \).

Labeled multi-Bernouilli RFS

A labeled multi-Bernouilli (LMB) RFS is a labeled version of the multi-Bernouilli RFS which, in turn, is a multitarget extension of the Bernouilli RFS. Recall that a Bernouilli RFS \( X \) on \( \mathbb{X} \) has probability \( r \) of being a singleton whose unique element is distributed on \( \mathbb{X} \) according to a suitable PDF \( s(\cdot) \), and probability \( q \triangleq 1 - r \) of being empty. Then, a multi-Bernouilli RFS \( X \) on \( \mathbb{X} \) is the union of a fixed number \( I \) of independent Bernouilli RFSs \( X^{(i)} \) with existence probability \( r^{(i)} \in (0, 1) \) and distributed on \( \mathbb{X} \) according to the PDF \( s^{(i)}(\cdot) \). An LMB RFS \( X \) with state space \( \mathcal{X} \) and label set \( \mathcal{L} \) is obtained from an unlabeled multi-Bernouilli RFS by appending distinct labels to the Bernouilli components.

**Definition 4** - Given \( r^{(i)} \in (0, 1) \) and PDFs \( s^{(i)}(\cdot) \) on \( \mathbb{X} \), for any \( \ell \in \mathcal{L} \), an RFS on \( \mathcal{F}(\mathbb{X} \times \mathcal{L}) \) with multiobject density

\[
f(X) = \Delta(X) \prod_{\ell \in \mathcal{L}(X)} r^{(\ell)} \prod_{\ell \notin \mathcal{L}(X)} [1 - r^{(\ell)}] \prod_{x, \ell \in \mathcal{X}} s^{(\ell)}(x) \tag{12}
\]

\[
\Delta(X) = \begin{cases} 1, & \text{if } |X| = |\mathcal{L}(X)| \\ 0, & \text{otherwise} \end{cases}
\]

is called LMB with labeled existence probabilities \( r^{(\ell)} \) and location PDFs \( s^{(\ell)}(\cdot) \).

III. MULTIOBJECT FUSION

The focus in this paper is on multiagent multitarget estimation. More precisely, multiple agents try to cooperatively estimate the multitarget of interest combining their own information. Let \( \mathcal{N} \) denote the finite set of agents and assume that, for each individual agent \( i \in \mathcal{N} \), a labeled (or unlabeled) multitarget density \( f_i(\cdot) \) (or \( f_{i,\ell}(\cdot) \)) is available. Then, a key issue is how to consistently fuse such multitarget densities taking into account that the agents may share common information and that such common information is impossible to single out. Hence, optimal (Bayes) fusion [30], [31] has to be ruled out and some robust suboptimal fusion approach has to be undertaken. In this respect, the paradigm of Kullback-Leibler fusion (average) has been successfully introduced in [10] for single-object PDFs and has been extended to unlabeled (CPHD) multitarget densities in [11].
Kullback-Leibler fusion

Let us first define the Kullback-Leibler divergence (distance) (KLD) between two (possibly labeled) multiobject densities \( f(X) \) and \( g(X) \) by

\[
D_{KL}(f \parallel g) \triangleq \int f(X) \log \frac{f(X)}{g(X)} \delta X
\]

(13)

where the integral in (13) must be interpreted as a set integral according to the definition \( \triangleq \). Then, the weighted Kullback-Leibler average \( \overline{f} \) of the agent multiobject densities \( f_i, i \in \mathcal{N} \), is defined as follows

\[
\overline{f} = \arg \inf_f \sum_{i \in \mathcal{N}} \omega_i \ D_{KL}(f \parallel f^i).
\]

(14)

with weights \( \omega_i \) satisfying

\[
\omega_i \geq 0, \quad \sum_{i \in \mathcal{N}} \omega_i = 1.
\]

(15)

Notice from (14) that the weighted KLA of the agent densities is the one that minimizes the weighted sum of distances from such densities. In particular, the choice \( \omega_i = 1/|\mathcal{N}| \) for any \( i \in \mathcal{N} \) in (14) provides the (unweighted) KLA which averages the agent densities giving to all of them the same level of confidence. An interesting interpretation of such a notion can be given recalling that, in Bayesian statistics, the KLD (13) can be seen as the information gain achieved when moving from a prior \( g(X) \) to a posterior \( f(X) \).

The following fundamental result holds.

**Theorem 1 (Kullback-Leibler fusion of general multiobject densities)** - The weighted KLA defined in (14) turns out to be given by

\[
\overline{f}(X) = \frac{\prod_{i \in \mathcal{N}} [f_i(X)]^{\omega_i}}{\prod_{i \in \mathcal{N}} [f_i(X)]^{\omega_i} \delta X}.
\]

(16)

**Proof:** The result for labeled multiobject densities can be proved in the same way as its unlabeled counterpart, i.e. Theorem 1 in [11], by replacing the set integral (5) with (11).

Notice that (16) states that the fused density \( \overline{f} \) is nothing but the normalized weighted geometric mean of the agent densities. Making use of the operators \( \oplus \) and \( \odot \) previously introduced, (16) can be more compactly rewritten as

\[
\overline{f} = \bigoplus_{i \in \mathcal{N}} (\omega_i \circ f_i).
\]

(17)

It must be pointed out that the fusion rule (16), which has been derived as Kullback-Leibler average of the local multiobject densities, coincides with the Generalized Covariance Intersection for multiobject fusion first proposed by Mahler [31] and is also called Exponential Mixture Density in [35]. Hereafter, the above result on Kullback-Leibler fusion of general multiobject densities will be specialized to PHD, CPHD and LMB representations in the following corollaries whose proofs will be omitted due to lack of space.

**Corollary 1 (Fusion of PHDs [11])** - The Kullback-Leibler fusion of agent Poisson RFSs with PHD functions \( d_i(x) = \pi_i s_i(x), i \in \mathcal{N} \), and fusion weights \( \omega_i \) satisfying (15), is a Poisson RFS with PHD function \( \overline{\pi}(x) = \pi \overline{s}(x) \), where

\[
\pi = \int \prod_{i \in \mathcal{N}} \pi_i s_i(x)^{\omega_i} dx
\]

(18)

\[
\overline{s}(x) = \prod_{i \in \mathcal{N}} s_i(x)^{\omega_i} = \bigoplus_{i \in \mathcal{N}} (\omega_i \circ s_i)(x)
\]

(19)

\[
\overline{\pi}(x) = \prod_{i \in \mathcal{N}} d_i(x)^{\omega_i}.
\]

(20)

**Corollary 2 (Fusion of CPHDs [11])** - The Kullback-Leibler fusion of agent id cluster RFSs with cardinality PMFs \( p_i(\cdot) \) and location PDFs \( s_i(\cdot), i \in \mathcal{N} \), and fusion weights \( \omega_i \) satisfying (15), is an id cluster RFS with location PDF \( \overline{\pi}(\cdot) \) given by (19) and cardinality PMF

\[
\overline{p}(n) = \frac{\prod_{i \in \mathcal{N}} p_i(n)^{\omega_i} \left\{ \int \prod_{i \in \mathcal{N}} s_i(x)^{\omega_i} dx \right\}^n}{\sum_{j=0}^{\infty} \prod_{i \in \mathcal{N}} p_i(j)^{\omega_i} \left\{ \int \prod_{i \in \mathcal{N}} s_i(x)^{\omega_i} dx \right\}^j}.
\]

(21)

**Corollary 3 (Fusion of LMBs [17])** - The Kullback-Leibler fusion of agent LMBs with existence probabilities \( \{r_i^{(\ell)}\}_{\ell \in L} \) and location PDFs \( \{s_i^{(\ell)}(\cdot)\}_{\ell \in L}, i \in \mathcal{N} \), and fusion weights \( \omega_i \) satisfying (15), is a LMB with existence probabilities \( \{\overline{\pi}^{(\ell)}(\cdot)\}_{\ell \in L} \) and location PDFs \( \{\overline{s}^{(\ell)}(\cdot)\}_{\ell \in L} \) given by

\[
\overline{\pi}^{(\ell)} = \int \prod_{i \in \mathcal{N}} r_i^{(\ell)} s_i^{(\ell)}(x)^{\omega_i} dx
\]

(22)

\[
\overline{s}^{(\ell)}(x) = \bigoplus_{i \in \mathcal{N}} (\omega_i \circ s_i^{(\ell)})(x).
\]

(23)

It is worth to notice that for all the considered representations (PHD, CPHD and LMB), the Kullback-Leibler fused (average) location PDFs are obtained by normalised weighted (by \( \omega_i \)) geometric averaging of the agent location PDFs. Conversely, the fused expected number of objects \( \overline{n} \) for PHD representations, cardinality PMF \( \overline{p}(\cdot) \) for CPHD representations and existence probabilities \( \overline{\pi}^{(\ell)}(\cdot) \) for LMB representations are given by more complicated expressions involving location PDFs as well.

IV. MULTIOBJECT CONSENSUS

From now on, it is assumed that agents are interconnected to form a network so that each agent can actually interchange multiobject information only with a subset of neighbours. The key issue to be addressed in this section is how to carry out
the multiobject fusion described in the previous section over
the network in a fully scalable, consistent and distributed way.

Network model

Let us consider a network of multiobject estimation agents
(nodes) as schematized in Fig. 1. The network consists of
heterogeneous and geographically dispersed nodes that have
processing, communication and sensing capabilities. More
specifically, each node can process local data as well as
exchange data with the neighbors and can get measurements
related to objects present in the surrounding environment.
The network of interest is characterized by the following features:
1) it has no central fusion node; 2) nodes are unaware of
the network topology, i.e. the number of nodes and their
connections.

The network can be described in terms of a directed graph
\( G = (\mathcal{N}, \mathcal{A}) \) where \( \mathcal{N} \) is the set of nodes (agents) and
\( \mathcal{A} \subseteq \mathcal{N} \times \mathcal{N} \) the set of arcs, representing links (connections)
between agents. In particular, \((i, j)\) belongs to \( \mathcal{A} \) if node
\( j \) can receive data from node \( i \). For each node \( j \in \mathcal{N}, \mathcal{N}_j \triangleq \{ i \in \mathcal{N} : (i, j) \in \mathcal{A} \} \)
denotes its set of in-neighbors, i.e. the set of nodes from which node \( j \) can receive data. By
definition, \((j, j) \in \mathcal{A} \) for any node \( j \in \mathcal{N} \) and, hence, \( j \in \mathcal{N}_j \)
for all \( j \).

Consensus

Consensus [1, 2] has emerged as a powerful tool for distribut-
ated computation (e.g. averaging, minimisation, maximisa-
tion, . . . ) over networks and has found widespread application in
distributed parameter/state estimation [3–10]. In essence, consensus
aims to perform a collective computation over a whole network by iterating, in each node \( i \) of the network, a
sequence of regional computations of the same type involving
the subnetwork \( \mathcal{N}_i \) of its in-neighbors.

In the context of this work, it is assumed that each node
\( i \) is provided with a local (labeled or unlabeled) multiobject
density \( f_i \) and wishes to compute, in a distributed and scalable
way, the collective Kullback-Leibler fusion
\[
\bar{f} = \bigoplus_{i \in \mathcal{N}} \left( \frac{1}{|\mathcal{N}|} \circ f_i \right) = \frac{1}{|\mathcal{N}|} \circ \left( \bigoplus_{i \in \mathcal{N}} f_i \right),
\] (24)
To this end, let \( \hat{f}_{i,0} = f_i \), then a consensus algorithm for the computation of (24) takes the iterative form
\[
\hat{f}_{i,k+1}(X) = \bigoplus_{j \in \mathcal{N}_i} \left( \omega_{i,j} \circ \hat{f}_{j,k}(X) \right), \quad \forall i \in \mathcal{N}
\] (25)
where the consensus weights must satisfy the conditions
\[
\omega_{i,j} \geq 0 \quad \forall i, j \in \mathcal{N}; \quad \sum_{j \in \mathcal{N}} \omega_{i,j} = 1 \quad \forall i \in \mathcal{N}.
\] (26)

In fact, thanks to the properties of the operators a)-f) listed in
[11, p. 513], it can be seen that
\[
\hat{f}_{i,k}(X) = \bigoplus_{j \in \mathcal{N}} \left( \omega_{i,j}^{(k)} \circ f_j(X) \right), \quad \forall i \in \mathcal{N}
\] (27)
where \( \omega_{i,j}^{(k)} \) is defined as the element \((i, j)\) of the matrix \( \Omega^k \)
and \( \Omega \) is the consensus matrix whose generic \((i, j)\)-element coincides with the consensus weight \( \omega_{i,j} \) (if \( j \notin \mathcal{N}_i \) then \( \omega_{i,j} \)

In [18] it has been proved that the Kullback-Leibler fusion
operator (27) takes the form
\[
\lim_{k \to +\infty} \omega_{i,j}^{(k)} = \frac{1}{|\mathcal{N}|}, \quad \forall i,j \in \mathcal{N}.
\]

Hence, as the number of consensus steps increases, each local
multiobject density “tends” to the collective KLA (24).

A necessary condition for the matrix \( \Omega \) to be primitive
is that the graph \( G \) is strongly connected [9]. In this case, a possible choice ensuring
convergence to the collective average for undirected graphs is
given by the so-called Metropolis weights [2, 9].

\[
\omega_{i,j} = \frac{1}{\max\{|\mathcal{N}_i|, |\mathcal{N}_j|\}} \quad i \in \mathcal{N}, j \in \mathcal{N}_i, i \neq j
\]
\[
\omega_{i,i} = 1 - \sum_{j \in \mathcal{N}_i, j \neq i} \omega_{i,j}.
\]

The consensus iteration (25) can clearly be specialised to
PHDs, CPHDs and LMBs making use of Corollaries 1-3 and
replacing in (18)-(23) the products over
PHDs, CPHDs and LMBs making use of Corollaries 1-3 and
the multiobject fusion described in the previous section over
the network in a fully scalable, consistent and distributed way.

Consensus

Consensus [1, 2] has emerged as a powerful tool for dis-
tributed computation (e.g. averaging, minimisation, maximisa-
tion, . . . ) over networks and has found widespread application in
distributed parameter/state estimation [3–10]. In essence, consensus
aims to perform a collective computation over a whole network by iterating, in each node \( i \) of the network, a
sequence of regional computations of the same type involving
the subnetwork \( \mathcal{N}_i \) of its in-neighbors.

In the context of this work, it is assumed that each node
\( i \) is provided with a local (labeled or unlabeled) multiobject
density \( f_i \) and wishes to compute, in a distributed and scalable
way, the collective Kullback-Leibler fusion
\[
\bar{f} = \bigoplus_{i \in \mathcal{N}} \left( \frac{1}{|\mathcal{N}|} \circ f_i \right) = \frac{1}{|\mathcal{N}|} \circ \left( \bigoplus_{i \in \mathcal{N}} f_i \right).
\] (24)
To this end, let \( \hat{f}_{i,0} = f_i \), then a consensus algorithm for the computation of (24) takes the iterative form
\[
\hat{f}_{i,k+1}(X) = \bigoplus_{j \in \mathcal{N}_i} \left( \omega_{i,j} \circ \hat{f}_{j,k}(X) \right), \quad \forall i \in \mathcal{N}
\] (25)
where the consensus weights must satisfy the conditions
\[
\omega_{i,j} \geq 0 \quad \forall i, j \in \mathcal{N}; \quad \sum_{j \in \mathcal{N}} \omega_{i,j} = 1 \quad \forall i \in \mathcal{N}.
\] (26)

In fact, thanks to the properties of the operators a)-f) listed in
[11, p. 513], it can be seen that
\[
\hat{f}_{i,k}(X) = \bigoplus_{j \in \mathcal{N}} \left( \omega_{i,j}^{(k)} \circ f_j(X) \right), \quad \forall i \in \mathcal{N}
\] (27)
where \( \omega_{i,j}^{(k)} \) is defined as the element \((i, j)\) of the matrix \( \Omega^k \)
and \( \Omega \) is the consensus matrix whose generic \((i, j)\)-element coincides with the consensus weight \( \omega_{i,j} \) (if \( j \notin \mathcal{N}_i \) then \( \omega_{i,j} \)

In [18] it has been proved that the Kullback-Leibler fusion
operator (27) takes the form
\[
\lim_{k \to +\infty} \omega_{i,j}^{(k)} = \frac{1}{|\mathcal{N}|}, \quad \forall i,j \in \mathcal{N}.
\]

Hence, as the number of consensus steps increases, each local
multiobject density “tends” to the collective KLA (24).

A necessary condition for the matrix \( \Omega \) to be primitive
is that the graph \( G \) associated with the sensor network be
strongly connected [9]. In this case, a possible choice ensuring
convergence to the collective average for undirected graphs is
given by the so-called Metropolis weights [2, 9].

\[
\omega_{i,j} = \frac{1}{\max\{|\mathcal{N}_i|, |\mathcal{N}_j|\}} \quad i \in \mathcal{N}, j \in \mathcal{N}_i, i \neq j
\]
\[
\omega_{i,i} = 1 - \sum_{j \in \mathcal{N}_i, j \neq i} \omega_{i,j}.
\]

The consensus iteration (25) can clearly be specialised to
PHDs, CPHDs and LMBs making use of Corollaries 1-3 and
replacing in (18)-(23) the products over
PHDs, CPHDs and LMBs making use of Corollaries 1-3 and
the multiobject fusion described in the previous section over
the network in a fully scalable, consistent and distributed way.
V. DISTRIBUTED MULTIOBJECT ESTIMATION

Combining Kullback-Leibler fusion and consensus with multiobject (labeled or unlabeled) filters, it is possible to develop effective and computationally feasible distributed multiobject estimation algorithms to be applied to, e.g., multitarget tracking, multirobot SLAM or multisource estimation. The general structure of such algorithms is outlined below. Different variants are clearly possible depending on the type of multiobject representation (e.g. PHD, CPHD, LMB) and on the type of multiobject filter implementation (e.g. Gaussian mixture or particle filter) being adopted.

Distributed multiobject estimation algorithm

At each time cycle $t \geq 1$, each agent $i \in \mathcal{N}$ carries out the following steps.

Step 1 - Local filtering: A multiobject (e.g. PHD or CPHD or LMB) filter updates the current local (PHD or CPHD or LMB) representation exploiting the multiobject time evolution model and the available sensor measurements.

Step 2 - Consensus: The multiobject representation of agent $i$ resulting from the local filtering step is repeatedly fused, for $K$ consensus iterations, with the ones from the neighbouring agents $j \in \mathcal{N}_i$.

Step 3 - Estimate extraction: estimates of the number of objects and of the relative states are suitably extracted from the fused multiobject representation resulting from the consensus step.

VI. MULTITARGET TRACKING CASE-STUDY

The present section reports a multitarget tracking case-study for the distributed multiobject estimation algorithm of Section V relying on the fusion rules of Corollaries 1-3. Three algorithms will be considered and referred to as Consensus PHD [11], Consensus CPHD [11] and Consensus LMB [17] filters, respectively, for the PHD, CPHD and LMB fusion.

A 2-dimensional tracking scenario consisting of 5 targets (depicted in Fig. 3) moving over a surveillance area of $50 \times 50 \, \text{m}^2$ is considered, wherein a sensor network of 4 range-only (Time Of Arrival, TOA) and 3 bearing-only (Direction Of Arrival, DOA) (see Fig. 2) is deployed.

Details about the local (non distributed) multiobject filters can be found in [29], [33] for PHD, [12], [13] for CPHD and respectively [14] for LMB. Please notice that, in principle, multiobject (but also single-object) representations are infinite-dimensional. Hence, for implementation purposes, finite-dimensional parametrizations of such representations need to be adopted. In this respect, the main problem is to finitely parameterize location PDFs as the support of the cardinality PMF for the CPHD representation can be restricted to a finite set by imposing a maximum number of objects while the label set $\mathcal{L}$ for the LMB representation is, by definition, discrete and finite. As for the location PDF, two finitely- parameterized representations based on the particle (Monte Carlo) or, respectively, Gaussian Mixture (GM) approaches are the most commonly employed. In [35]- [36], a Monte Carlo implementation of a distributed PHD filter has been presented. For distributed multiobject estimation over a network, however, the limited processing power and energy resources of the individual agents seem to suggest the more parsimonious GM approach, as usually the number of involved Gaussian components is orders of magnitude lower than the number of particles required for a satisfactory estimation performance. Motivated by this consideration, a GM implementation of a consensus-based distributed CPHD filter has been proposed in [11]. As pointed out in [11], the Kullback-Leibler fusion of GMs is no longer a GM due to exponentiation. Hence, to preserve the GM form of the various location PDFs involved in the (PHD, CPHD or LMB) distributed multiobject estimation algorithms, a suitable approximation of the GM exponentiation, suggested in [34] and already used in [11], can be adopted.

The kinematic object state is denoted by $x = [p_x, \dot{p}_x, p_y, \dot{p}_y]^T$, i.e. the planar position and velocity. The motion of objects is modeled by the filters according to the Nearly-Constant Velocity (NCV) model [19]-[22]:

$$ x_{k+1} = \begin{bmatrix} 1 & T_s & 0 & 0 \\ 0 & 1 & T_s & 0 \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + w_k $$

where $w_k$ has zero-mean and variance

$$ Q = \sigma_w^2 \begin{bmatrix} \frac{1}{2}T_s^4 & \frac{1}{2}T_s^5 & 0 & 0 \\ \frac{1}{2}T_s^3 & \frac{1}{2}T_s^4 & T_s & 0 \\ 0 & 0 & \frac{1}{2}T_s^3 & \frac{1}{2}T_s^4 \\ 0 & 0 & T_s & \frac{1}{2}T_s^3 \end{bmatrix} $$

with $\sigma_w = 5 \, \text{[m/s]}$ and sampling interval $T_s = 5 \, \text{s}$.

The TOA and DOA sensors are characterized by the following measurement functions:

$$ h^i(x) = \begin{cases} \angle[(p_x - x_i) + j \, (p_y - y_i)], & \text{DOA} \\ \sqrt{(p_x - x_i)^2 + (p_y - y_i)^2}, & \text{TOA} \end{cases} $$

Fig. 2. Network with 7 sensors: 4 TOA and 3 DOA.

Fig. 3. Target trajectories considered in the simulation experiment. The start/end point for each trajectory is denoted, respectively, by ■ and ⋆. The ■ indicates a rendezvous point.
where \((x_i, y_i)\) represents the known position of sensor \(i\). The standard deviation of the measurement noises are taken respectively as \(\sigma_{DOA} = 1^\circ\) and \(\sigma_{TOA} = 100[m]\). The *Unscented Kalman Filter* (UKF) \([37]\) is used in each sensor to cope with the non-linearity of the sensor measurement functions.

The clutter is assumed, for each sensor, as a Poisson RFS with an average intensity of \(\lambda_c = 5\) and a uniform spatial distribution over the surveillance area. The probability of object detection is \(P_D = 0.99\).

Incomplete prior information for target birth locations is assumed and is modeled according to a 10-component LMB RFS \(f_B = \{ (r^{(i)}, p_B^{(i)}) \}_i \). Table I gives a detailed summary of such components.

The *Optimal SubPattern Assignment* (OSPA) metric \([38]\) with Euclidean distance, \(p = 2\), and cutoff \(c = 600[m]\) is used to evaluate the performance of the distributed multiobject filters. The reported metric is averaged over 100 Monte Carlo trials for the same target trajectories but different, independently generated, clutter and measurement noise realizations. The duration of each simulation trial is fixed to 1000 [s] (200 samples).

A single consensus step \(K = 1\) is employed for all the simulations.

Figs. 4 and 5 display the statistics (mean and standard deviation) of the estimated number of targets obtained, respectively, with the Consensus CPHD and the Consensus LMB filters. Such distributed algorithms estimate the object cardinality accurately. Note that the difficulties introduced by the rendezvous point (e.g. merged or lost tracks) are correctly tackled by both (Consensus CPHD and LMB) distributed algorithms. Conversely, in this case-study, the Consensus PHD filter failed to achieve satisfactory performance compared to Consensus CPHD and LMB filters. For this reason, results obtained with the Consensus PHD filter are not reported.

Fig. 6 shows the OSPA metric. The more accurate localization of the Consensus LMB filter can be attributed to two factors: (a) the “spooky effect” \([39]\) causes the Consensus CPHD filter to temporarily drop targets which are subjected to missed detections and to declare multiple estimates for existing tracks in place of the dropped targets, and (b) the Consensus LMB filter is generally able to better localize objects due to a more accurate propagation of the posterior density.

### Table I

**Components of the LMB RFS birth process at a given time \(k\).**

<table>
<thead>
<tr>
<th>Label (m_B^{(i)})</th>
<th>((k, 1))</th>
<th>((k, 2))</th>
<th>((k, 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0), (40000), (0) (\uparrow)</td>
<td>(0), (25000), (0) (\uparrow)</td>
<td>(0), (5000), (0) (\uparrow)</td>
<td></td>
</tr>
<tr>
<td>(0), (0), (40000) (\uparrow)</td>
<td>(0), (0), (25000) (\uparrow)</td>
<td>(0), (0), (5000) (\uparrow)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Label (m_B^{(i)})</th>
<th>((k, 4))</th>
<th>((k, 5))</th>
<th>((k, 6))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5000), (0), (0) (\uparrow)</td>
<td>(25000), (0), (0) (\uparrow)</td>
<td>(36000), (0), (0) (\uparrow)</td>
<td></td>
</tr>
<tr>
<td>(0), (0), (0) (\uparrow)</td>
<td>(0), (0), (0) (\uparrow)</td>
<td>(0), (0), (0) (\uparrow)</td>
<td></td>
</tr>
</tbody>
</table>

The OSPA distance \(c = 600[m]\), \(p = 2\). Fig. 6 shows the OSPA metric. The more accurate localization of the Consensus LMB filter can be attributed to two factors: (a) the “spooky effect” \([39]\) causes the Consensus CPHD filter to temporarily drop targets which are subjected to missed detections and to declare multiple estimates for existing tracks in place of the dropped targets, and (b) the Consensus LMB filter is generally able to better localize objects due to a more accurate propagation of the posterior density.

### VII. Conclusions

The paper has reviewed the concepts of Kullback-Leibler fusion and consensus for both labeled and unlabeled RFSs and their application to scalable distributed multiagent multiobject estimation over a sensor network.

Possible topics for future work are 1) to consider sensors with different and non-uniform field-of-view; 2) to compare the proposed Gaussian mixture implementation of the consensus multiobject filters with a particle filter implementation; 3) to apply multiobject consensus filters to multirobot SLAM and to estimation of multiple diffusive sources.
ACKNOWLEDGMENTS

This work was partially supported by a grant of Selex ES.

REFERENCES


