# Average Kullback-Leibler Divergence for Random Finite Sets


Giorgio Battistelli, Luigi Chisci and Claudio Fantacci
Dipartimento di Ingegneria dell'Informazione
Università di Firenze
Firenze 50139, Italy
e-mail: giorgio.battistelli@unifi.it
e-mail: luigi.chisci@unifi.it
e-mail: claudio.fantacci@unifi.it

Alfonso Farina
Selex ES
(retired at the end of 2014)
Rome 00131, Italy
e-mail: alfonso.farina@outlook.it

Ba-Ngu Vo
Department of Electrical and Computer Engineering
Curtin University
Bentley, WA 6102, Australia
e-mail: ba-ngu.vo@curtin.edu.au



***Abstract*—The paper deals with the fusion of multiobject information over a network of heterogeneous and geographically dispersed nodes with sensing, communication and processing capabilities. To exploit the benefits of sensor networks for multiobject estimation problems, like e.g. multitarget tracking and multirobot SLAM (Simultaneous Localization and Mapping), a key issue to be addressed is how to consistently fuse (average) locally updated multiobject densities. In this paper we discuss the generalization of Kullback-Leibler average, originally conceived for single-object densities (i.e. probability density functions) to (both unlabeled and labeled) multiobject densities. Then, with a view to develop scalable and reliable distributed multiobject estimation algorithms, we review approaches to iteratively compute, in each node of the network, the collective multiobject average via scalable and neighborwise computations.**



***Index Terms*—multiobject estimation; sensor networks; distributed fusion; random finite sets; consensus; multitarget tracking.**


## I. Introduction

The recent breakthrough in wireless sensor technology has opened up the possibility to develop efficient surveillance/monitoring systems consisting of radio interconnections of a multitude of low cost and low energy consumption devices with sensing, communication and processing capabilities. To best exploit this emerging technology, it is however fundamental to redesign multiobject estimation algorithms taking into account the following issues: 1) the single node has limited sensing and computational capabilities and must also limit data transmission, which is primarily responsible for energy consumption; 2) processing must be carried out in a distributed fashion with no coordination of a central unit and in a scalable way with respect to the network size; 3) each node is unaware of the correlations existing between its own information and the information received from other nodes.

In recent years, *consensus* has emerged as a powerful tool for distributed computation over networks [1], [2] and has been widely used in distributed parameter/state estimation algorithms [3]–[10]. In particular, a consensus algorithm can be used for distributed averaging over a network; each node (agent) aims to compute the collective average of a given quantity by iterative regional averages, where the terms *collective* and *regional* mean *over all network nodes* and, respectively, *over neighboring nodes only*. Central to consensus for distributed state estimation is the notion of Kullback-Leibler average for probability density functions [10]. An important functionality of a surveilance/monitoring sensor network is *multiobject estimation* (i.e. the joint detection and state estimation of multiple objects of interest in a given area under surveillance/monitoring) for which consensus on multiobject information among the different nodes of the network becomes a key issue.

In this paper, we provide an overview of Kullback-Leibler average for multiobject probability density functions. In particular, we review the generalization of the Kullback-Leibler average to multiobject probability densities proposed in [11], and how this concept is exploited to develop an effective distributed multiobject estimation algorithm based on the Gaussian Mixture Cardinalized Probability Hypothesis Density (GM-CPHD) filter [12], [13]. We also review the recent application of Kullback-Leibler fusion for multiobject densities developed in [11] to the recently introduced labeled random finite set models [14]–[16], which have lead to more accurate distributed multitarget tracking algorithms proposed in [17].

The rest of the paper is organized as follows. Section II introduces the necessary background for modelling multiobjects as random finite sets and reviews different (both unlabeled and labeled) multiobject representations. Section III recalls from [11] the notion of Kullback-Leibler average (fusion) of multiobject distributions and, in particular, provides new analytical expressions for the Kullback-Leibler fusion of certain labeled multiobject densities. Section IV shows how consensus can be exploited in order to fuse, in a scalable fashion, multiobject densities over a peer-to-peer (coordination-free) network. Section V outlines the structure of a general distributed multiobject estimation algorithm based on Kullback-Leibler fusion and consensus. Section VI shows a case-study concerning the application of Kullback-Leibler fusion and consensus to multitarget tracking. Finally section

VII provides concluding remarks and perspectives for future work.

## II. MULTIOBJECT REPRESENTATION

*Notation*

Throughout the paper, the following notation will be adopted. $\mathbb{R}, \mathbb{R}_+, \mathbb{N}$ denote the sets of real, nonnegative real and, respectively, positive integer numbers. $E[\cdot]$ denotes the *expectation* operator. $|X|$ denotes the *cardinality* (number of elements) of the finite set $X$. Given two real-valued functions $f$ and $g$ defined over the same domain, their *inner product* is defined as $< f, g > \stackrel{\triangle}{=} \int f\, g$ where the integral (possibly a *set integral* to be defined later) is extended to the whole domain. Then, the operators $\oplus$ and $\odot$ [10], [11], [18] are defined as follows:

$$\begin{aligned} f \oplus g &\stackrel{\triangle}{=} f\, g < f, g >^{-1} \\ \omega \odot f &\stackrel{\triangle}{=} f^{\omega} < f^{\omega}, 1 >^{-1} \end{aligned}$$

for any $\omega > 0$.

*Multiobject estimation*

*Multiobject estimation* aims to jointly detect an unknown number of objects of interest in a given area of competence and to estimate their states. This has relevant applications, for instance, in *multitarget tracking* [19]–[23], *SLAM (Simultaneous Localization and Mapping)* [24] and *multisource estimation* [25] wherein the objects of interest are respectively targets moving in the surveillance area, relevant elements (landmarks) of the environment surrounding a navigating robot or vehicle, sources diffusing heat or pollutants in a monitored area. Since the multiobject to be estimated is characterised by a twofold randomness in both the number of objects and in their states, a natural approach that will be pursued in this paper is to represent it in terms of a *Random Finite Set (RFS)*.

*Random finite sets*

An RFS $X \subset \mathbb{X}$ is a random variable taking values in $\mathcal{F}(\mathbb{X})$, the collection of all finite subsets of the single-object state space $\mathbb{X}$. While $\mathcal{F}(\mathbb{X})$ does not inherit the usual Euclidean notion of probability density from $\mathbb{X}$, a measure-theoretic notion of probability density on $\mathcal{F}(\mathbb{X})$ is available [26]. However, we adopt the *Finite Set Statistic* (FISST) notion of density since it is convenient and by-passes measure theoretic constructs [27], [28]. Hereafter, the basic concepts of FISST needed for the subsequent developments will be briefly reviewed.

An RFS $X$ is completely characterized by its *multiobject density*. Multiobject densities of RFSs are defined with respect to the reference measure $\mu$ given by

$$\mu(\mathcal{T}) = \sum_{i=0}^{\infty} \frac{1}{i! K^i} \int_{\mathbb{X}^i} 1_{\mathcal{T}}(\{x_1, \ldots, x_i\}) d(x_1, \ldots, x_i) \quad (1)$$

for any (measurable) subset $\mathcal{T}$ of $\mathcal{F}(\mathbb{X})$. The measure $\mu$ is analogous to the Lebesgue measure on $\mathbb{X}$ (indeed it is the unnormalized distribution of a Poisson RFS with unit intensity $u = 1/K$ when the state space $\mathbb{X}$ is bounded). Moreover, it was shown in [26] that for this choice of reference measure, the integral of a function $\pi : \mathcal{F}(\mathbb{X}) \to \mathbb{R}$, given by

$$\int \pi(X) \mu(dX) = \sum_{i=0}^{\infty} \frac{1}{i! K^i} \int_{\mathbb{X}^i} \pi(\{x_1, \ldots, x_i\}) d(x_1, \ldots, x_i), \quad (2)$$

is equivalent to Mahler's set integral [28], which is defined for a generic function $g(\cdot)$ as follows:

$$\int_S g(X)\, \delta X \stackrel{\triangle}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} g(\{x_1, \ldots, x_n\})\, d(x_1, \ldots, x_n)\,. \quad (3)$$

In particular,

$$\beta(S) \stackrel{\triangle}{=} Prob\,(X \subset S) = \int_S f(X)\, \delta X$$

gives the probability that the RFS $X$ is included in the subset $S$ of $\mathbb{X}$, and $f(\cdot)$ is called the FISST density. Note that while the FISST density $f(\cdot)$ is not a probability density, it is equivalent to the multiobject probability $\pi(\cdot)$ as shown in [26]. This result is key to the generalization of concepts involving probability density to multioject FISST densities. Hence, in this work, we use the FISST density as a probability density.

The first-order moment of the multiobject density, better known as *Probability Hypothesis Density* (PHD) or intensity function, has been found to be a very successful characterization [28]. In order to define the PHD function, let us introduce the number of elements of the RFS $X$ within $S \subseteq \mathbb{X}$ which is clearly given by

$$n(S) = \int_S \sum_{\xi \in X} \delta_\xi(x) dx$$

where $\delta_\xi(\cdot)$ is the Dirac delta centered at $\xi$. The PHD function is defined such that the expected number of elements of $X$ in $S$ is obtained by

$$E[n(S)] = \int_S d(x)\, dx\,. \quad (4)$$

Without loss of generality, the PHD function can be expressed as

$$d(x) = \overline{n}\; s(x) \quad (5)$$

where

$$\begin{aligned} \overline{n} &= E[n] = E[n(\mathbb{X})] = \sum_{n=0}^{\infty} n p(n) \quad (6) \\ s(x) &= d(x) / \overline{n}\,. \quad (7) \end{aligned}$$

are respectively the expected number of objects and a single-object PDF, and $p(n)$ denotes the *cardinality* PMF (Probability Mass Function) i.e. the probability that the RFS $X$ have $n$ elements.

*Poisson and iid cluster RFSs*

Hereafter, two commonly used unlabeled representations of multiobjects, i.e. Poisson and iid cluster RFSs (processes) will be reviewed. A Poisson RFS is uniquely characterised by its intensity function $d(\cdot)$ as follows.

**Definition 1** - Given a function $d(\cdot) : \mathbb{X} \to \mathbb{R}_+$, an RFS on $\mathcal{F}(\mathbb{X})$ with multiobject density

$$f(X) \; = \; e^{-\int d(x) dx} \prod_{x \in X} d(x) \; = \; e^{-\overline{n}} \, \overline{n}^{|X|} \prod_{x \in X} s(x) \quad (8)$$

is called *Poisson process* with PHD function (intensity) $d(\cdot)$, or equivalently *expected number of objects* $\overline{n} = \int d(x) dx$ and location PDF $s(\cdot) = d(\cdot)/\overline{n}$. ■

An iid (independent identically distributed) cluster RFS is completely characterised by its intensity function $d(\cdot)$ and cardinality PMF $p(n)$ as follows.

**Definition 2** - Given a PMF $p(n)$ on the nonnegative integers and a PDF $s(\cdot)$ on $\mathbb{X}$, an RFS on $\mathcal{F}(\mathbb{X})$ with multiobject density

$$f(X) \; = \; |X|! \, p(|X|) \prod_{x \in X} s(x) \quad (9)$$

is called *iid cluster process* with cardinality PMF $p(\cdot)$ and location PDF $s(\cdot)$, or equivalently PHD function $d(\cdot) = \overline{n} s(\cdot)$ where $\overline{n} = \sum_{n=0}^{\infty} n p(n)$. ■

Comparing (8) and (9), it is clear that a Poisson RFS is nothing but a special case of iid cluster RFS wherein the number of objects is restricted to be Poisson-distributed with parameter $\overline{n}$, i.e. $p(n) \; = \; e^{-\overline{n}} \, \overline{n}^n \, / \, n!$. Assuming that the multitarget RFSs are Poisson or iid cluster processes is at the basis of the PHD [29] or, respectively, *Cardinalized* PHD (CPHD) [12] filtering approaches to multiobject estimation. For this reason, Poisson and iid cluster RFSs will be also referred to in the sequel as PHD and, respectively, CPHD representations of multiobjects.

*Labeled random finite sets*

In certain applications of multiobject estimation (e.g. multitarget tracking) the aim is not only to estimate the number and the states of the objects, but also to keep track of their trajectories over time. To this end, the notion of label is introduced in the RFS framework [15]- [16] so that each object can be uniquely identified and its track be reconstructed. Let $\mathbb{L} = \{\ell_i : i \in \mathbb{N}\}$ be the discrete *label set*. To incorporate object identity, a label $\ell \in \mathbb{L}$ is appended to the state $x \in \mathbb{X}$ of each object and a multiobject is accordingly regarded as an RFS on the *labeled state space* $\mathbb{X} \times \mathbb{L}$, i.e. a *labeled state* $\mathbf{x} = (x, \ell) \in \mathbb{X} \times \mathbb{L}$ is assigned to each object. Let $\mathcal{L} : \mathbb{X} \times \mathbb{L} \to \mathbb{L}$ be the projector of the labeled state space into the label set so that $\mathcal{L}((x, \ell)) = \ell$. Then, to avoid situations in which multiple objects have the same label, the following definition of *labeled RFS* is introduced.

**Definition 3** - A labeled RFS $\mathbf{X}$ with state space $\mathbb{X}$ and label set $\mathbb{L}$ is an RFS on $\mathbb{X} \times \mathbb{L}$ such that any realization satisfies

$$|\mathbf{X}| \; = \; |\mathcal{L}(\mathbf{X})| \, . \quad (10)$$

Notice that the condition (10) imposes that all elements of $\mathbf{X}$ have distinct labels. The set integral (3) is extended to any function $\mathbf{g} : \mathcal{F}(\mathbb{X} \times \mathbb{L}) \to \mathbb{R}$ defined on a labeled RFS, as follows:

$$\int \mathbf{g}(\mathbf{X}) \, \delta \mathbf{X} \stackrel{\triangle}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\ell_1, \ldots, \ell_n \in \mathbb{L}} \int \mathbf{g}(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\}) \, d(x_1, \ldots, x_n) \, . \quad (11)$$

Hereinafter labeled states, spaces, multiobject densities will be denoted by bold symbols, e.g. $\mathbf{x}$, $\mathbf{X}$, $\mathbf{f}(\cdot)$ instead of their unlabelled counterparts $x$, $X$, $f(\cdot)$.

*Labeled multi-Bernouilli RFS*

A *labeled multi-Bernouilli* (LMB) RFS is a labeled version of the multi-Bernouilli RFS which, in turn, is a multiobject extension of the Bernouilli RFS. Recall that a Bernouilli RFS $X$ on $\mathbb{X}$ has probability $r$ of being a singleton whose unique element is distributed on $\mathbb{X}$ according to a suitable PDF $s(\cdot)$, and probability $q \stackrel{\triangle}{=} 1 - r$ of being empty. Then, a multi-Bernouilli RFS $X$ on $\mathbb{X}$ is the union of a fixed number $I$ of independent Bernouilli RFSs $X^{(i)}$ with *existence probability* $r^{(i)} \in (0, 1)$ and distributed on $\mathbb{X}$ according to the PDF $s^{(i)}(\cdot)$. An LMB RFS $\mathbf{X}$ with state space $\mathbb{X}$ and label set $\mathbb{L}$ is obtained from an unlabeled multi-Bernouilli RFS by appending distinct labels to the Bernouilli components.

**Definition 4** - Given $r^{(\ell)} \in (0, 1)$ and PDFs $s^{(\ell)}(\cdot)$ on $\mathbb{X}$, for any $\ell \in \mathbb{L}$, an RFS on $\mathcal{F}(\mathbb{X} \times \mathbb{L})$ with multiobject density

$$\mathbf{f}(\mathbf{X}) = \Delta(\mathbf{X}) \prod_{\ell \in \mathcal{L}(\mathbf{X})} r^{(\ell)} \prod_{\ell \in \mathbb{L} \setminus \mathcal{L}(\mathbf{X})} \left[ 1 - r^{(\ell)} \right] \prod_{(x, \ell) \in \mathbf{X}} s^{(\ell)}(x) \quad (12)$$

$$\Delta(\mathbf{X}) = \begin{cases} 1, & \text{if } |\mathbf{X}| \; = \; |\mathcal{L}(\mathbf{X})| \\ 0, & \text{otherwise} \end{cases}$$

is called *LMB* with labeled existence probabilities $r^{(\ell)}$ and location PDFs $s^{\ell}(\cdot)$. ■

## III. Multiobject fusion

The focus in this paper is on *multiagent* multiobejct estimation. More precisely, multiple agents try to cooperatively estimate the multiobject of interest combining their own information. Let $\mathcal{N}$ denote the finite set of agents and assume that, for each individual agent $i \in \mathcal{N}$, a labeled (or unlabeled) multiobject density $\mathbf{f}_i(\cdot)$ (or $f_i(\cdot)$) be available. Then, a key issue is how to consistently fuse such multiobject densities taking into account that the agents may share common information and that such common information is impossible to single out. Hence, optimal (Bayes) fusion [30], [31] has to be ruled out and some robust suboptimal fusion approach has to be undertaken. In this respect, the paradigm of Kullback-Leibler fusion (average) has been successfully introduced in [10] for single-object PDFs and has been extended to unlabeled (CPHD) multiobject densities in [11].

*Kullback-Leibler fusion*

Let us first define the *Kullback-Leibler divergence (distance)* (KLD) between two (possibly labeled) multiobject densities $\mathbf{f}(\mathbf{X})$ and $\mathbf{g}(\mathbf{X})$ by

$$D_{KL}(\mathbf{f} \parallel \mathbf{g}) \triangleq \int \mathbf{f}(\mathbf{X}) \, \log \frac{\mathbf{f}(\mathbf{X})}{\mathbf{g}(\mathbf{X})} \, \delta\mathbf{X} \tag{13}$$

where the integral in (13) must be interpreted as a set integral according to the definition (11). Then, the weighted *Kullback-Leibler average* (KLA) $\overline{\mathbf{f}}$ of the agent multiobject densities $\mathbf{f}_i$, $i \in \mathcal{N}$, is defined as follows

$$\overline{\mathbf{f}} = \arg\inf_{\mathbf{f}} \sum_{i \in \mathcal{N}} \omega_i \, D_{KL}\left(\mathbf{f} \parallel \mathbf{f}^i\right). \tag{14}$$

with weights $\omega_i$ satisfying

$$\omega_i \geq 0, \qquad \sum_{i \in \mathcal{N}} \omega_i = 1. \tag{15}$$

Notice from (14) that the weighted KLA of the agent densities is the one that minimizes the weighted sum of distances from such densities. In particular, the choice $\omega_i = 1/|\mathcal{N}|$ for any $i \in \mathcal{N}$ in (14) provides the (unweighted) KLA which averages the agent densities giving to all of them the same level of confidence. An interesting interpretation of such a notion can be given recalling that, in Bayesian statistics, the KLD (13) can be seen as the information gain achieved when moving from a prior $\mathbf{g}(\mathbf{X})$ to a posterior $\mathbf{f}(\mathbf{X})$.

The following fundamental result holds.

**Theorem 1** (*Kullback-Leibler fusion of general multiobject densities*) - The weighted KLA defined in (14) turns out to be given by

$$\overline{\mathbf{f}}(\mathbf{X}) = \frac{\prod_{i \in \mathcal{N}} \left[\mathbf{f}_i(\mathbf{X})\right]^{\omega_i}}{\int \prod_{i \in \mathcal{N}} \left[\mathbf{f}_i(\mathbf{X})\right]^{\omega_i} \, \delta\mathbf{X}}. \tag{16}$$

*Proof:* The result for labeled multiobject densities can be proved in the same way as its unlabeled counterpart, i.e. Theorem 1 in [11], by replacing the set integral (3) with (11).

Notice that (16) states that the fused density $\mathbf{f}$ is nothing but the normalized weighted geometric mean of the agent densities. Making use of the operators $\oplus$ and $\odot$ previously introduced, (16) can be more compactly rewritten as

$$\overline{\mathbf{f}} = \bigoplus_{i \in \mathcal{N}} \left(\omega_i \odot \mathbf{f}_i\right). \tag{17}$$

It must be pointed out that the fusion rule (16), which has been derived as Kullback-Leibler average of the local multi-object densities, coincides with the *Generalized Covariance Intersection* for multiobject fusion first proposed by Mahler [31] and is also called *Exponential Mixture Density* in [35]. Hereafter, the above result on Kullback-Leibler fusion of general multiobject densities will be specialized to PHD, CPHD and LMB representations in the following corollaries whose proofs will be omitted due to lack of space.

**Corollary 1** (*Fusion of PHDs* [11]) - The Kullback-Leibler fusion of agent Poisson RFSs with PHD functions $d_i(x) = \overline{n}_i s_i(x)$, $i \in \mathcal{N}$, and fusion weights $\omega_i$ satisfying (15), is a Poisson RFS with PHD function $\overline{d}(x) = \overline{n}\,\overline{s}(x)$, where

$$\overline{n} = \int \prod_{i \in \mathcal{N}} \left[\overline{n}_i \, s_i(x)\right]^{\omega_i} dx \tag{18}$$

$$\overline{s}(x) = \frac{\prod_{i \in \mathcal{N}} s_i(x)^{\omega_i}}{\int \prod_{i \in \mathcal{N}} s_i(x)^{\omega_i} \, dx} = \bigoplus_{i \in \mathcal{N}} \left(\omega_i \odot s_i\right)(x) \tag{19}$$

$$\overline{d}(x) = \prod_{i \in \mathcal{N}} d_i(x)^{\omega_i}. \tag{20}$$

**Corollary 2** (*Fusion of CPHDs* [11]) - The Kullback-Leibler fusion of agent iid cluster RFSs with cardinality PMFs $p_i(\cdot)$ and location PDFs $s_i(\cdot)$, $i \in \mathcal{N}$, and fusion weights $\omega_i$ satisfying (15), is an iid cluster RFS with location PDF $\overline{s}(\cdot)$ given by (19) and cardinality PMF

$$\overline{p}(n) = \frac{\prod_{i \in \mathcal{N}} p_i(n)^{\omega_i} \left\{\int \prod_{i \in \mathcal{N}} s_i(x)^{\omega_i} dx\right\}^n}{\sum_{j=0}^{\infty} \prod_{i \in \mathcal{N}} p_i(j)^{\omega_i} \left\{\int \prod_{i \in \mathcal{N}} s_i(x)^{\omega_i} dx\right\}^j}. \tag{21}$$

**Corollary 3** (*Fusion of LMBs* [17]) - The Kullback-Leibler fusion of agent LMBs with existence probabilities $\left\{r_i^{(\ell)}\right\}_{\ell \in \mathbb{L}}$ and location PDFs $\left\{s_i^{(\ell)}(\cdot)\right\}_{\ell \in \mathbb{L}}$, $i \in \mathcal{N}$, and fusion weights $\omega_i$ satisfying (15), is an LMB with existence probabilities $\left\{\overline{r}^{(\ell)}\right\}_{\ell \in \mathbb{L}}$ and location PDFs $\left\{\overline{s}^{(\ell)}(x)\right\}_{\ell \in \mathbb{L}}$ given by

$$\overline{r}^{(\ell)} = \frac{\int \prod_{i \in \mathcal{N}} \left[r_i^{(\ell)} s_i^{(\ell)}(x)\right]^{\omega_i} dx}{\prod_{i \in \mathcal{N}} \left(1 - r_i^{(\ell)}\right)^{\omega_i} + \int \prod_{i \in \mathcal{N}} \left[r_i^{(\ell)} s_i^{(\ell)}(x)\right]^{\omega_i} dx} \tag{22}$$

$$\overline{s}^{(\ell)}(x) = \bigoplus_{i \in \mathcal{N}} \left(\omega_i \odot s_i^{(\ell)}\right)(x). \tag{23}$$

It is worth to notice that for all the considered representations (PHD, CPHD and LMB), the Kullback-Leibler fused (average) location PDFs are obtained by normalised weighted (by $\omega_i$) geometric averaging of the agent location PDFs. Conversely, the fused expected number of objects $\overline{n}$ for PHD representations, cardinality PMF $\overline{p}(\cdot)$ for CPHD representations and existence probabilities $\overline{r}^{(\ell))}$ for LMB representations are given by more complicated expressions involving location PDFs as well.

## IV. Multiobject consensus

From now on, it is assumed that agents are interconnected to form a network so that each agent can actually interchange multiobject information only with a subset of neighbours. The key issue to be addressed in this section is how to carry out the multiobject fusion described in the previous section over the network in a fully scalable, consistent and distributed way.

*Network model*

Let us consider a network of multiobject estimation agents (nodes) as schematized in Fig. 1. The network consists of

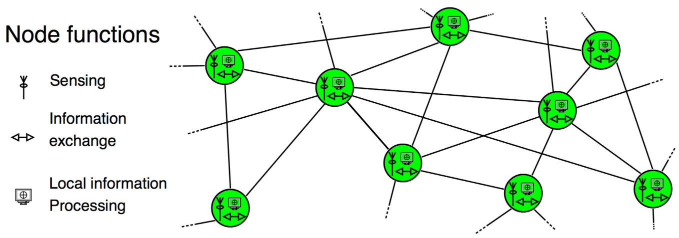


Fig. 1. Network model

heterogeneous and geographically dispersed nodes that have processing, communication and sensing capabilities. More specifically, each node can process local data as well as exchange data with the neighbors and can get measurements related to objects present in the surrounding environment. The network of interest is characterized by the following features: 1) it has no *central fusion node*; 2) nodes are unaware of the network topology, i.e. the number of nodes and their connections.

The network can be described in terms of a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is the set of nodes (agents) and $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$ the set of arcs, representing links (connections) between agents. In particular, $(i, j)$ belongs to $\mathcal{A}$ if node $j$ can receive data from node $i$. For each node $j \in \mathcal{N}$, $\mathcal{N}_j \stackrel{\triangle}{=} \{i \in \mathcal{N} : (i, j) \in \mathcal{A}\}$ denotes its set of in-neighbors, i.e. the set of nodes from which node $j$ can receive data. By definition, $(j, j) \in \mathcal{A}$ for any node $j \in \mathcal{N}$ and, hence, $j \in \mathcal{N}_j$ for all $j$.

*Consensus*

*Consensus* [1], [2] has emerged as a powerful tool for distributed computation (e.g. averaging, minimisation, maximisation, . . . ) over networks and has found widespread application in distributed parameter/state estimation [3]–[10]. In essence, consensus aims to perform a collective computation over a whole network by iterating, in each node $i$ of the network, a sequence of regional computations of the same type involving the subnetwork $\mathcal{N}_i$ of its in-neighbors.

In the context of this work, it is assumed that each node $i$ is provided with a local (labeled or unlabelled) multiobject density $\mathbf{f}_i$ and wishes to compute, in a distributed and scalable way, the collective Kullback-Leibler fusion

$$\overline{\mathbf{f}} = \bigoplus_{i \in \mathcal{N}} \left( \frac{1}{|\mathcal{N}|} \odot \mathbf{f}_i \right) = \frac{1}{|\mathcal{N}|} \odot \left( \bigoplus_{i \in \mathcal{N}} \mathbf{f}_i \right). \quad (24)$$

To this end, let $\hat{\mathbf{f}}_{i,0} = \mathbf{f}_i$, then a consensus algorithm for the computation of (24) takes the iterative form

$$\hat{\mathbf{f}}_{i,k+1}(\mathbf{X}) = \bigoplus_{j \in \mathcal{N}_i} \left( \omega_{i,j} \odot \hat{\mathbf{f}}_{j,k}(\mathbf{X}) \right), \qquad \forall i \in \mathcal{N} \quad (25)$$

where the *consensus weights* must satisfy the conditions

$$\omega_{i,j} \geq 0 \quad \forall i, j \in \mathcal{N}; \qquad \sum_{j \in \mathcal{N}_i} \omega_{i,j} = 1 \quad \forall i \in \mathcal{N}. \quad (26)$$

In fact, thanks to the properties of the operators a)-f) listed in [11, p. 513], it can be seen that

$$\hat{\mathbf{f}}_{i,k}(\mathbf{X}) = \bigoplus_{j \in \mathcal{N}} \left( \omega_{i,j}^{(k)} \odot \mathbf{f}_j(\mathbf{X}) \right), \qquad \forall i \in \mathcal{N} \quad (27)$$

where $\omega_{i,j}^{(k)}$ is defined as the element $(i, j)$ of the matrix $\Omega^k$ and $\Omega$ is the consensus matrix whose generic $(i, j)$-element coincides with the consensus weight $\omega_{i,j}$ (if $j \notin \mathcal{N}_i$ then $\omega_{i,j}$ is taken as 0). In this respect, it is well known that if $\Omega$ is primitive (i.e. there exists an integer $m$ such that all entries of $\Omega^m$ are strictly positive) and doubly stochastic (i.e. all it rows and columns sum up to one), one has

$$\lim_{k \to +\infty} \omega_{i,j}^{(k)} = \frac{1}{|\mathcal{N}|}, \quad \forall i, j \in \mathcal{N}.$$

Hence, as the number of consensus steps increases, each local multiobject density "tends" to the collective KLA (24).

A necessary condition for the matrix $\Omega$ to be primitive is that the graph $\mathcal{G}$ associated with the sensor network be strongly connected [9]. In this case, a possible choice ensuring convergence to the collective average for undirected graphs is given by the so-called *Metropolis weights* [2], [9].

$$\begin{aligned} \omega_{i,j} &= \frac{1}{\max\{|\mathcal{N}_i|, |\mathcal{N}_j|\}}, \quad i \in \mathcal{N}, j \in \mathcal{N}_i, i \neq j \\ \omega_{i,i} &= 1 - \sum_{j \in \mathcal{N}_i, j \neq i} \omega_{i,j}. \end{aligned}$$

The consensus iteration (25) can clearly be specialised to PHDs, CPHDs and LMBs making use of Corollaries 1-3 and replacing in (18)-(23) the products over $\mathcal{N}$ with exponential weights $\omega_i$ with products over $\mathcal{N}_i$ with exponential weights $\omega_{i,j}$, to be carried out in each node $i$. For instance, the LMB consensus iteration at node $i$ takes the form

$$r_{i,k+1}^{(\ell)} = \frac{\displaystyle\int \prod_{j \in \mathcal{N}_i} \left[ r_{j,k}^{(\ell)} s_{j,k}^{(\ell)}(x) \right]^{\omega_{i,j}} dx}{\displaystyle\prod_{j \in \mathcal{N}_i} \left( 1 - r_{j,k}^{(\ell)} \right)^{\omega_{i,j}} + \int \prod_{j \in \mathcal{N}_i} \left[ r_{j,k}^{(\ell)} s_{j,k}^{(\ell)}(x) \right]^{\omega_{i,j}} dx}$$

$$s_{i,k+1}^{(\ell)}(x) = \bigoplus_{j \in \mathcal{N}_i} \left( \omega_{i,j} \odot s_{j,k}^{(\ell)} \right)(x)$$

which, under the initialisation $r_{i,0}^{(\ell)} = r_i^{(\ell)}$ and $s_{i,0}^{(\ell)}(\cdot) = s_i^{(\ell)}(\cdot)$, converges to $\overline{r}^{(\ell)}$ and $\overline{s}^{(\ell)}(\cdot)$ in (22)-(23) (with $\omega_i = |\mathcal{N}|^{-1}$ for all $i$) as $k \to \infty$ for any label $\ell \in \mathbb{L}$ and any agent $i \in \mathcal{N}$ provided that the consensus weights $\omega_{i,j}$ are suitably chosen. PHD and CPHD consensus iterations, which are omitted due to lack of space, can be obtained in a similar way.

In [18] it has been proved that the Kullback-Leibler fusion guarantees immunity to double counting of information and that, further, the consensus approach always give rise to multiobject densities which avoid double counting irrespectively of the number of consensus iterations being carried out.

## V. Distributed multiobject estimation

Combining Kullback-Leibler fusion and consensus with multiobject (labeled or unlabeled) filters, it is possible to develop effective and computationally feasible distributed multiobject estimation algorithms to be applied to, e.g., multitarget tracking, multirobot SLAM or multisource estimation. The general structure of such algorithms is outlined below. Different variants are clearly possible depending on the type of multiobject representation (e.g. PHD, CPHD, LMB) and on the type of multiobject filter implementation (e.g. Gaussian mixture or particle filter) being adopted.

**Distributed multiobject estimation algorithm**

At each time cycle $t \geq 1$, each agent $i \in \mathcal{N}$ carries out the following steps.
**Step 1 - Local filtering**: A multiobject (e.g. PHD or CPHD or LMB) filter updates the current local (PHD or CPHD or LMB) representation exploiting the multiobject time evolution model and the available sensor measurements.
**Step 2 - Consensus**: The multiobject representation of agent $i$ resulting from the local filtering step is repeatedly fused, for $K$ consensus iterations, with the ones from the neighbouring agents $j \in \mathcal{N}_i$.
**Step 3 - Estimate extraction**: estimates of the number of objects and of the relative states are suitably extracted from the fused multiobject representation resulting from the consensus step. ■

Details about the local (non distributed) multiobject filters can be found in [29], [33] for PHD, [12], [13] for CPHD and respectively [14] for LMB. Please notice that, in principle, multiobject (but also single-object) representations are infinite-dimensional. Hence, for implementation purposes, finite-dimensional parametrizations of such representations need to be adopted. In this respect, the main problem is to finitely parameterize location PDFs as the support of the cardinality PMF for the CPHD representation can be restricted to a finite set by imposing a maximum number of objects while the label set $\mathbb{L}$ for the LMB representation is, by definition, discrete and finite. As for the location PDF, two finitely-parameterized representations based on the *particle* (*Monte Carlo*) or, respectively, *Gaussian Mixture (GM)* approaches are the most commonly employed. In [35]- [36], a Monte Carlo implementation of a distributed PHD filter has been presented. For distributed multiobject estimation over a network, however, the limited processing power and energy resources of the individual agents seem to suggest the more parsimonious GM approach, as usually the number of involved Gaussian components is orders of magnitude lower than the number of particles required for a satisfactory estimation performance. Motivated by this consideration, a GM implementation of a consensus-based distributed CPHD filter has been proposed in [11]. As pointed out in [11], the Kullback-Leibler fusion of GMs is no longer a GM due to exponentiation. Hence, to preserve the GM form of the various location PDFs involved in the (PHD, CPHD or LMB) distributed multiobject estimation algorithms, a suitable approximation of the GM exponentiation, suggested in [34] and already used in [11], can be adopted.

## VI. Multitarget tracking case-study

The present section reports a multitarget tracking case-study for the distributed multiobject estimation algorithm of Section V relying on the fusion rules of Corollaries 1-3. Three algorithms will be considered and referred to as Consensus PHD [11], Consensus CPHD [11] and Consensus LMB [17] filters, respectively, for the PHD, CPHD and LMB fusion.

A 2-dimensional tracking scenario consisting of 5 targets (depicted in Fig. 3) moving over a surveillance area of $50 \times 50\,[km^2]$ is considered, wherein a sensor network of 4 *range-only* (Time Of Arrival, TOA) and 3 *bearing-only* (Direction Of Arrival, DOA) (see Fig. 2) is deployed.

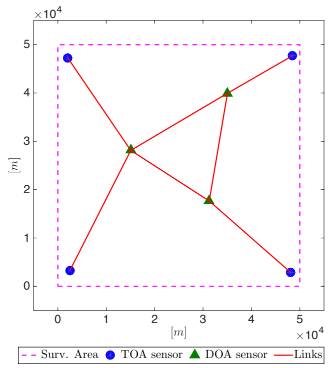


Fig. 2. Network with 7 sensors: 4 TOA and 3 DOA.

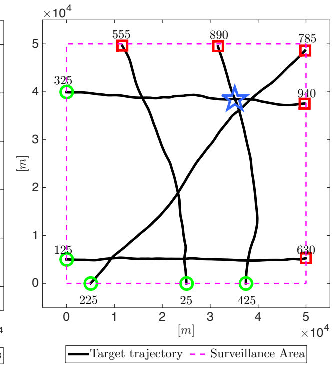


Fig. 3. Target trajectories considered in the simulation experiment. The start/end point for each trajectory is denoted, respectively, by •\■. The ★ indicates a rendezvous point.

The kinematic object state is denoted by $x = [p_x,\, \dot{p}_x,\, p_y,\, \dot{p}_y]^\top$, i.e. the planar position and velocity. The motion of objects is modeled by the filters according to the Nearly-Constant Velocity (NCV) model [19]–[22]:

$$x_{k+1} = \begin{bmatrix} 1 & T_s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + w_k$$

where $w_k$ has zero-mean and variance

$$Q = \sigma_w^2 \begin{bmatrix} \frac{1}{4}T_s^4 & \frac{1}{2}T_s^3 & 0 & 0 \\ \frac{1}{2}T_s^3 & T_s^2 & 0 & 0 \\ 0 & 0 & \frac{1}{4}T_s^4 & \frac{1}{2}T_s^3 \\ 0 & 0 & \frac{1}{2}T_s^3 & T_s^2 \end{bmatrix}$$

with $\sigma_w = 5\,[m/s^2]$ and sampling interval $T_s = 5\,[s]$.

The TOA and DOA sensors are characterized by the following measurement functions:

$$h^i(x) = \begin{cases} \angle[(p_x - x^i) + j\,(p_y - y^i)], & \text{DOA} \\ \sqrt{(p_x - x^i)^2 + (p_y - y^i)^2}, & \text{TOA} \end{cases}$$

where $(x^i, y^i)$ represents the known position of sensor $i$. The standard deviation of the measurement noises are taken respectively as $\sigma_{DOA} = 1\,[°]$ and $\sigma_{TOA} = 100\,[m]$. The *Unscented Kalman Filter* (UKF) [37] is used in each sensor to cope with the non linearity of the sensor measurement functions.

The clutter is assumed, for each sensor, as a Poisson RFS with an average intensity of $\lambda_c = 5$ and a uniform spatial distribution over the surveillance area. The probability of object detection is $P_D = 0.99$.

Incomplete prior information for target birth locations is assumed and is modeled according to a 10-component LMB RFS $\mathbf{f}_B = \left\{\left(r_B^{(\ell)}, p_B^{(\ell)}\right)\right\}_{\ell \in \mathbb{B}}$. Table I gives a detailed summary of such components.

TABLE I
COMPONENTS OF THE LMB RFS BIRTH PROCESS AT A GIVEN TIME $k$.

$$r^{(\ell)} = 0.09$$
$$p_B^{(\ell)}(x) = \mathcal{N}\left(x;\, m_B^{(\ell)}, P_B\right)$$
$$P_B = \text{diag}\left(10^6, 10^4, 10^6, 10^4\right)$$

| **Label** | $(k, 1)$ | $(k, 2)$ | $(k, 3)$ |
|---|---|---|---|
| $m_B^{(\ell)}$ | $[0,\ 0,\ 40000,\ 0]^\top$ | $[0,\ 0,\ 25000,\ 0]^\top$ | $[0,\ 0,\ 5000,\ 0]^\top$ |

| **Label** | $(k, 4)$ | $(k, 5)$ | $(k, 6)$ |
|---|---|---|---|
| $m_B^{(\ell)}$ | $[5000,\ 0,\ 0,\ 0]^\top$ | $[25000,\ 0,\ 0,\ 0]^\top$ | $[36000,\ 0,\ 0,\ 0]^\top$ |

| **Label** | $(k, 7)$ | $(k, 8)$ |
|---|---|---|
| $m_B^{(\ell)}$ | $[50000,\ 0,\ 15000,\ 0]^\top$ | $[50000,\ 0,\ 40000,\ 0]^\top$ |

| **Label** | $(k, 9)$ | $(k, 10)$ |
|---|---|---|
| $m_B^{(\ell)}$ | $[40000,\ 0,\ 50000,\ 0]^\top$ | $[10000,\ 0,\ 50000,\ 0]^\top$ |

The *Optimal SubPattern Assignment* (OSPA) metric [38] with Euclidean distance, $p = 2$, and cutoff $c = 600\,[m]$ is used to evaluate the performance of the distributed multiobject filters. The reported metric is averaged over 100 Monte Carlo trials for the same target trajectories but different, independently generated, clutter and measurement noise realizations. The duration of each simulation trial is fixed to $1000\,[s]$ (200 samples).

A single consensus step $K = 1$ is employed for all the simulations.

Figs. 4 and 5 display the statistics (mean and standard deviation) of the estimated number of targets obtained, respectively, with the Consensus CPHD and the Consensus LMB filters. Such distributed algorithms estimate the object cardinality accurately. Note that the difficulties introduced by the rendezvous point (e.g. merged or lost tracks) are correctly tackled by both (Consensus CPHD and LMB) distributed algorithms. Conversely, in this case-study, the Consensus PHD filter failed to achieve satisfactory performance compared to Consensus CPHD and LMB filters. For this reason, results obtained with the Consensus PHD filter are not reported.

Fig. 6 shows the OSPA metric. The more accurate localization of the Consensus LMB filter can be attributed to two factors: (a) the "spooky effect" [39] causes the Consensus CPHD filter to temporarily drop targets which are subjected to missed detections and to declare multiple estimates for existing tracks in place of the dropped targets, and (b) the Consensus LMB filter is generally able to better localize objects due to a more accurate propagation of the posterior density.

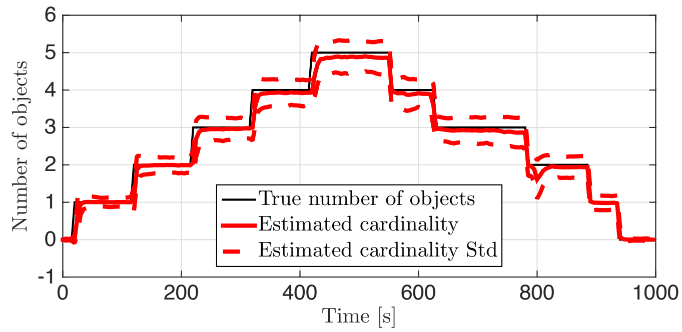


Fig. 4. Cardinality statistics for the Consensus CPHD filter under high SNR.

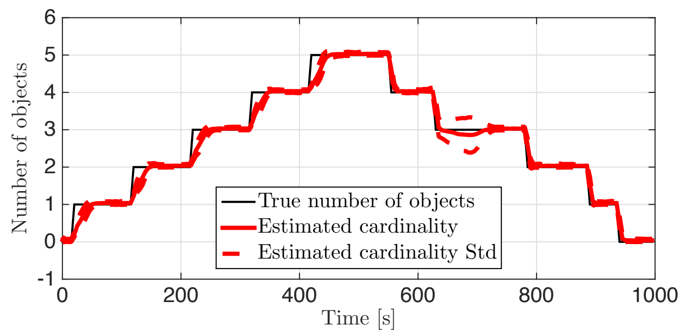


Fig. 5. Cardinality statistics for the consensus CLMB filter under high SNR.

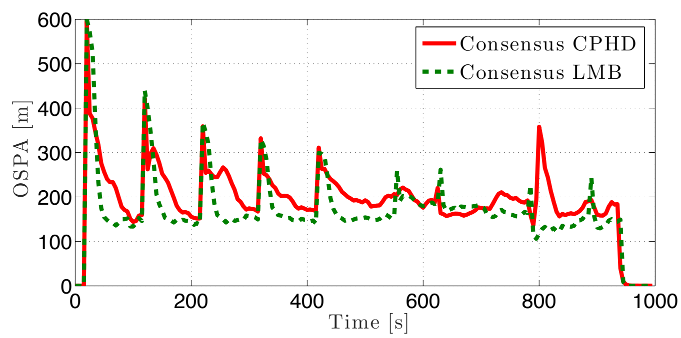


Fig. 6. OSPA distance ($c = 600\,[m]$, $p = 2$) under high SNR.

## VII. CONCLUSIONS

The paper has reviewed the concepts of Kullback-Leibler fusion and consensus for both labeled and unlabeled RFSs and their application to scalable distributed multiagent multiobject estimation over a sensor network.

Possible topics for future work are 1) to consider sensors with different and non-uniform field-of-view; 2) to compare the proposed Gaussian mixture implementation of the consensus multiobject filters with a particle filter implementation; 3) to apply multiobject consensus filters to multirobot SLAM and to estimation of multiple diffusive sources.

## Acknowledgments

This work was partially supported by a grant of Selex ES.

## References

[1] R. Olfati-Saber, J.A. Fax, and R.M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 49-54, 2007.

[2] L. Xiao, L., S. Boyd and S. Lall, "A scheme for robust distributed sensor fusion based on average consensus," *Proc. 4th Int. Symposium on Information Processing in Sensor Networks (IPSN)*, pp. 63-70, 2005.

[3] R. Olfati-Saber, "Distributed Kalman filtering for sensor networks," *Proc. 46th IEEE Conf. on Decision and Control*, pp. 5492-5498, 2007.

[4] M. Kamgarpour and C. Tomlin, "Convergence properties of a decentralized Kalman filter," *Proc. 47th IEEE Conf. on Decision and Control*, pp. 3205-3210, 2008.

[5] R. Carli, A. Chiuso, L. Schenato, and S. Zampieri, "Distributed Kalman filtering based on consensus strategies," *IEEE Journal on Selected Areas in Communications*, vol. 26, pp. 622-633, 2008.

[6] S. S. Stankovic, M.S. Stankovic, and D.M. Stipanovic, "Consensus based overlapping decentralized estimation with missing observations and communication faults," *Automatica*, vol. 45, no. 6, pp. 1397-1406, 2009.

[7] F. S. Cattivelli and A.H. Sayed, "Diffusion strategies for distributed Kalman filtering and smoothing," *IEEE Trans. on Automatic Control*, vol. 55, pp. 2069-2084, 2010.

[8] M. Farina, G. Ferrari-Trecate and R. Scattolini, "Distributed moving horizon estimation for linear constrained systems," *IEEE Trans. on Automatic Control*, vol. 55, no. 11, pp. 2462-2475, 2010.

[9] G. C. Calafiore and F. Abrate, "Distributed linear estimation over sensor networks," *International Journal of Control*, vol. 82, no. 5, pp. 868-882, 2009.

[10] G. Battistelli, and L. Chisci, "Kullback-Leibler average, consensus on probability densities, and distributed state estimation with guaranteed stability," *Automatica*, vol. 50, no. 3, pp. 707-718, 2014.

[11] G. Battistelli, L. Chisci, C. Fantacci, A. Farina and A. Graziano, "Consensus CPHD filter for distributed multitarget tracking," *IEEE Journal of Selected Topics in Signal Processing*, vol. 7, no. 3, pp. 508-520, 2013.

[12] R. P. S. Mahler, "PHD filters of higher order in target number," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 43, no. 4, pp. 1523-1543, 2007.

[13] B.-T. Vo, B.-N. Vo and A. Cantoni, "Analytic implementations of the cardinalized probability hypothesis density filter," *IEEE Trans. on Signal Processing*, vol. 55, no. 7, pp. 3553-3567, 2007.

[14] S. Reuter, B.-T. Vo, B.-N. Vo and K. Dietmayer, "The labeled multi-Bernouilli filter", *IEEE Trans. on Signal Processing*, vol. 62, no. 12, pp. 3246-3260, 2014.

[15] B.-T. Vo and B.-N. Vo, "Labeled random finite sets and multi-object conjugate priors," *IEEE Trans. on Signal Processing*, vol. 61, no. 13, pp. 3460-3475, 2013.

[16] B.-N. Vo, B.-T. Vo and D. Phung, "Labeled random finite sets and the Bayes multi-target tracking filter," *IEEE Trans. on Signal Processing*, vol. 62, no. 24, pp. 6554-6567, 2014.

[17] C. Fantacci, B.-N. Vo, B.-T. Vo, G. Battistelli and L. Chisci, "Consensus labeled random finite set filtering for distributed multi-object tracking," *arXiv preprint arXiv:1501.01579*, 2014.

[18] G. Battistelli, L. Chisci, C. Fantacci, A. Farina and R. Mahler, "Distributed fusion of multitarget densities and consensus PHD/CPHD filters", *Proc. SPIE Defense, Security and Sensing*, vol. 9474, Baltimore, MD, 2015.

[19] A. Farina and F.A. Studer, *Radar data processing, vol. I: introduction and tracking*. John Wiley, 1985.

[20] A. Farina and F.A. Studer, *Radar data processing, vol. II: advanced topics and applications*. Researches Studies Press, England, 1986.

[21] Y. Bar-Shalom and T.E. Fortmann, *Tracking and data association*. Academic Press, San Diego, 1988.

[22] Y. Bar-Shalom and X.R. Li, *Multitarget-multisensor tracking: principles and techniques*. YBS Publishing, Storrs, CT, 1995.

[23] S. Blackman and R. Popoli, *Design and analysis of modern tracking systems*. Artech House, Norwood, MA, 1999.

[24] M. Adams, B.-N. Vo, R. Mahler and J. Mullane, "SLAM gets a PHD", *IEEE Robotics and Automation Magazine*, vol. 21, no. 2, pp. 26-37, 2014.

[25] J. Weimer, B. Sinopoli and B.H. Krogh, "Multiple source detection and localisation in advection-diffusion processes using wireless sensor networks," *Proc. 30th IEEE Real-Time Systems Symposium*, pp. 333-342, Washington, DC, USA, 2009.

[26] B.-N. Vo, S. Singh and A. Doucet, "Sequential Monte Carlo methods for multi-target filtering with random finite sets," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 41, no. 4, pp. 1224-1245, 2005.

[27] I. Goodman, R.P.S. Mahler and H. Nguyen, *Mathematics of data fusion*. Kluwer Academic Publishers, Norwell, MA, USA, 1997.

[28] R. P. S. Mahler, *Statistical multisource-multitarget information fusion*. Artech House, Norwood, MA, 2007.

[29] R. P. S. Mahler, "Multitarget Bayes filtering via first-order multitarget moments," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 39, no. 4, pp. 1152-1178, 2003.

[30] C.-Y. Chong, S. Mori and K.-C. Chang, "Distributed multitarget multi-sensor tracking," in *Multitarget-Multisensor Tracking: Advanced Applications*, Y. Bar-Shalom Ed., Artech House, 1990, ch. 8.

[31] R.P.S. Mahler, "Optimal/robust distributed data fusion: a unified approach," *Proc. of the SPIE Defense and Security Symposium 2000*, 2000.

[32] D.E. Clark, S.J. Julier, R.P.S. Mahler and B. Ristić, "Robust multi-object fusion with unknown correlation," *Proc. of the 2010 Sensor Signal Processing for Defense Conference (SSPD 2010)*, London, United Kingdom, 2010.

[33] B.-N. Vo and W.K. Ma, "The Gaussian mixture probability hypothesis density filter," *IEEE Trans. on Signal Processing*, vol. 54, no. 11, pp. 4091-4104, 2006.

[34] S. J. Julier, "An empirical study into the use of Chernoff information for robust, distributed fusion of Gaussian mixture models," *Proc. 9th Int. Conf. on Information Fusion*, pp. 1-8, Florence, Italy, 2006.

[35] M. Uney, S. Julier, D.E. Clark and B. Ristić, "Monte Carlo realisation of a distributed multi-object fusion algorithm", *Sensor Signal Processing for Defence (SSPD)*, 2010.

[36] M. Uney, D.E. Clark and S.J. Julier, "Distributed fusion of PHD filters via exponential mixture densities", *IEEE Journal of Selected Topics in Signal Processing*, vol. 7, no. 3, pp. 521-531, 2013.

[37] S. J. Julier and J. K. Uhlmann, "Unscented filtering and nonlinear estimation," *Proc. of the IEEE*, vol. 92, pp. 401-422, 2004.

[38] D. Schuhmacher, B.-T. Vo and B.-N. Vo, "A consistent metric for performance evaluation of multi-object filters," *IEEE Trans. on Signal Processing*, vol. 56, no. 8, pp. 3447-3457, 2008.

[39] D. Fränken, M. Schmidt and M. Ulmke, "Spooky action at a distance in the cardinalized probability hypothesis density filter", *IEEE Trans. on Aerospace and Electronic Systems*, vol. 45, no. 4, pp. 1657-1664, 2009.