A Partially Uniform Target Birth Model for Gaussian Mixture PHD/CPHD Filtering

Michael Beard, Ba Tuong Vo, Ba-Ngu Vo, and Sanjeev Arulampalam

Abstract

The conventional GMPHD/CPHD filters require the PHD for target births to be a Gaussian mixture, which is potentially inefficient because careful selection of the mixture parameters may be required to ensure good performance. Here we present approximations which allow part of the birth PHD to be uniformly distributed, obviating the need to use large Gaussian mixtures to model target births. The benefits of this approach are demonstrated by simulations on a bearings-only filtering scenario.

Index Terms

Multi-target tracking, Bearings-only tracking, Gaussian mixture probability hypothesis density filter, Target birth model

I. INTRODUCTION

The Probability Hypothesis Density (PHD) filter [1, 2] is an algorithm for tracking multiple targets in the presence of clutter and missed detections. It was developed as a computationally tractable approximation to the full multi-target Bayes recursion, which is formulated using the theory of Random Finite Sets (RFS). The PHD is a function which characterises the intensity of targets at all points in the single-target state space, and it is therefore often referred to as the intensity function. The PHD filter propagates the intensity function in time, and corrects it based on sets of target observations. An intuitive interpretation of the PHD/CPHD filter in terms of occupancy bins can be found in [5].

Since the PHD filter only propagates the first moment, a great deal of information about the multi-target density is discarded. To improve on this, the Cardinalised PHD (CPHD) filter was developed in [2, 7]. The CPHD filter

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also propagates the PHD, but in addition to this it propagates a probability distribution on the number of targets, known as the cardinality distribution. This means that the Poisson assumption for the number of targets is relaxed, leading to less variance in the the estimated number of targets.

There are two methods of implementing the PHD/CPHD filter, one based on sequential Monte Carlo (SMC) [3], and another based on Gaussian mixtures (GM) [4, 20]. The original forms of the Gaussian mixture PHD/CPHD filters assume that both the posterior PHD at the previous time, and the prior PHD for new targets entering the scene at the current time, are both Gaussian mixtures [4, 8]. Although it is possible to approximate an arbitrary density using a sum of Gaussians, this approach is potentially inefficient since a large number of Gaussians may be necessary to obtain a reasonable approximation. This is especially true in the case where the birth density is uniform, which applies in the common situation where targets may appear anywhere with equal probability.

The use of a diffuse target birth model is addressed for the GMPHD filter in [9] and for the SMC-PHD filter in [11]. A more formal derivation of the modified GMPHD filter, and the derivation of a modified GMCPHD filter was given in [13], along with some preliminary simulation results for a simple linear-Gaussian scenario. In this paper, we extend the work in [13] by applying the modified GMPHD/CPHD filters to the more challenging non-linear bearings-only filtering problem, allowing us to model the angular component of the PHD for new targets using a uniform distribution across the entire bearing space. We compare the modified filters to the corresponding traditional versions under various GM approximations to a birth model with uniform bearing.

The modified filter equations are based on a principled approximation, which is derived by showing that the birth PHD can be defined using a uniform distribution on the part of the state space which is directly observed through the measurements, and a Gaussian distribution for the unobserved parts. In this paper, we refer to this as a Partially Uniform Birth (PUB) model, and the filters using this model as the PUB-GMPHD and PUB-GMCPHD filters. Use of the PUB model leads to a predicted PHD which is no longer a Gaussian mixture. Nonetheless, the updated PHD can be accurately approximated by a Gaussian mixture, provided that the birth PHD is Gaussian in the unobserved state components, and the uniform distribution for the observed state components covers a large enough region such that the effect of truncating the Gaussians is negligible.

The main advantage of this approach is a reduced number of parameters that must be selected by the end user. To construct a GM representation of the birth PHD requires selecting a number of parameters, including the mean locations, covariance matrices and weights of the Gaussians. Poor selection can adversely affect performance, so there is a potentially significant benefit in removing this requirement. Furthermore, simulations show that this method provides much better performance than a GM with a small number of components, and similar performance to a GM with a large number of components.

II. GMPHD Filter with Partially Uniform Target Birth Model (PUB-GMPHD)

The general form of the PHD filter recursion (without target spawning) is given by [1]

$$v_{k|k-1}(x) = \int P_{S,k}(\zeta) f_{k|k-1}(x|\zeta) v_{k-1}(\zeta) d\zeta + \gamma_k(x) \tag{1}$$
$$v_k(x) = (1 - P_{D,k}(x)) v_{k|k-1}(x) + \sum_{z \in Z} \frac{P_{D,k}(x) g_k(z|x) v_{k|k-1}(x)}{\kappa_k(z) + \int P_{D,k}(\xi) g_k(z|\xi) v_{k|k-1}(\xi) d\xi}$$

where v_{k-1} is the posterior PHD from time $k-1$, $P_{S,k}(\zeta)$ is the probability of survival at time k for a target in state ζ, $f_{k|k-1}$ is the target transition density from time $k-1$ to time k, $\gamma_k(x)$ is the prior PHD of spontaneous target births at time k, $P_{D,k}(x)$ is the probability of detection at time k for a target in state x, $g_k(z|x)$ is the likelihood function for a measurement z given target state x, and $\kappa_k(z)$ is the clutter density at time k in the vicinity of measurement z. We now derive approximations of the PHD prediction and update for the case when the PHD of spontaneous target births is partially uniform, which we refer to as the PUB-GMCPHD filter.

A. Prediction

Let the posterior PHD at time $k-1$ be a Gaussian mixture of the form

$$v_{k-1}(x) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(x; m_{k-1}^{(i)}, P_{k-1}^{(i)}) \tag{2}$$

Let θ represent the part of the state x that is directly observed through measurement z, and ϕ be the part of the state which is not observed (i.e. θ is either a scalar or a vector consisting of the terms involved in the measurement likelihood calculation (9), and ϕ is a scalar or vector consisting of the remaining terms). The PHD in (2) may now be expressed as

$$v_{k-1}(\theta, \phi) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(\theta, \phi; \theta_{k-1}^{(i)}, \phi_{k-1}^{(i)}, \Omega_{k-1}^{(i)}) \tag{3}$$

where $\theta_{k-1}^{(i)}$ is the observed part of $m_{k-1}^{(i)}$, $\phi_{k-1}^{(i)}$ is the unobserved part of $m_{k-1}^{(i)}$, and $\Omega_{k-1}^{(i)}$ is the covariance matrix $P_{k-1}^{(i)}$ transformed into the space defined by (θ, ϕ). In the remainder of this paper, we abbreviate $\theta_k^{(i)}, \phi_k^{(i)}$ to $(\theta, \phi)_k^{(i)}$. We now define the PHD of spontaneous target births as

$$\gamma_k(\theta, \phi) = w_k^b U(\theta; \mathcal{B}) \mathcal{N}(\phi; \bar{\phi}, \sigma_\phi^2) \tag{4}$$

where w_k^b is the expected number of targets appearing at time k, $U(\theta; \mathcal{B})$ is the uniform distribution in θ over the region $\mathcal{B}$, $\bar{\phi}$ is the prior mean of the unmeasured state component, and σ_ϕ^2 is its prior variance. Now, the predicted PHD at time k is given by

$$v_{k|k-1}(\theta, \phi) = v_{S,k|k-1}(\theta, \phi) + \gamma_k(\theta, \phi) \tag{5}$$

where

$$v_{S,k|k-1}(\theta, \phi) = P_{S,k} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(\theta, \phi; (\theta, \phi)_{k|k-1}^{(i)}, \Omega_{k|k-1}^{(i)}) \tag{6}$$

assuming the process noise is additive Gaussian, and the survival probability $P_{S,k}$ is independent of target state.

Let us augment the state with a binary variable β so we may distinguish between the surviving components and the birth component, resulting in the following form for the predicted PHD

$$v_{k|k-1}(\theta,\phi,\beta) = \begin{cases} \sum_{i=1}^{J_{k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)},\Omega_{k|k-1}^{(i)}) & ,\beta=0 \\ w_k^b U(\theta;\mathcal{B})\mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2) & ,\beta=1 \end{cases} \tag{7}$$

where the predicted weights are $w_{k|k-1}^{(i)} = P_{S,k} w_{k-1}^{(i)}$. As noted in [11], separating the surviving and birth components using the β variable is necessary in order to avoid biasing the cardinality estimates.

B. Update

Let us assume that new targets are always detected at their time of birth. For simplicity, we also assume that the detection probability for surviving targets is independent of their state, which leads to the definition

$$P_{D,k}(\theta,\phi,\beta) = \begin{cases} P_{D,k} & ,\beta=0 \\ 1 & ,\beta=1 \end{cases} \tag{8}$$

Furthermore, we assume that the measurement likelihood function is Gaussian and defined by

$$g(z|\theta,\phi,\beta) = \mathcal{N}(z;\theta,\sigma_\theta^2) \tag{9}$$

The posterior intensity is now given by (10) and (11), where, for the sake of brevity, the subscript $k|k-1$ is omitted for $w^{(i)}$, $\theta^{(i)}$, $(\theta,\phi)^{(i)}$ and $\Omega^{(i)}$.

$$\begin{aligned} v_k(\theta,\phi,0) = (1-P_{D,k}) &\sum_{i=1}^{J_{k-1}} w^{(i)}\mathcal{N}(\theta,\phi;(\theta,\phi)^{(i)},\Omega^{(i)}) \\ &+ \sum_{z\in Z}\sum_{i=1}^{J_{k-1}} \frac{P_{D,k}\mathcal{N}(z;\theta^{(i)},\sigma_\theta^2)w^{(i)}\mathcal{N}(\theta,\phi;(\theta,\phi)^{(i)},\Omega^{(i)})}{\kappa_k(z)+\int P_{D,k}(\xi)g(z|\xi)v_{k|k-1}(\xi)d\xi} \end{aligned} \tag{10}$$

$$v_k(\theta,\phi,1) = \sum_{z\in Z} \frac{\mathcal{N}(\theta;z,\sigma_\theta^2)w_k^b U(\theta;\mathcal{B})\mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2)}{\kappa_k(z)+\int P_{D,k}(\xi)g(z|\xi)v_{k|k-1}(\xi)d\xi} \tag{11}$$

On the denominator in (10) and (11) we have the following (see appendix for derivation),

$$\begin{aligned} \int P_{D,k}(\xi)g(z|\xi)v_{k|k-1}(\xi)d\xi = &\sum_{i=1}^{J_{k-1}} P_{D,k} w_{k|k-1}^{(i)} q_k^{(i)}(z) \\ &+ w_k^b \int \mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2)d\phi \int \mathcal{N}(\theta;z,\sigma_\theta^2)\frac{1_{\mathcal{B}}(\theta)}{V_{\mathcal{B}}}d\theta \end{aligned} \tag{12}$$

where $q_k^{(i)}(z) = \mathcal{N}(z;\theta_{k|k-1}^{(i)},S_{k|k-1}^{(i)})$ is the likelihood of measurement z against component i (with $S_{k|k-1}^{(i)}$ being the innovation covariance for component i), $1_{\mathcal{B}}(\theta)$ is the indicator function for the region $\mathcal{B}$, and $V_{\mathcal{B}}$ is the volume of region $\mathcal{B}$. We now make the approximation

$$\mathcal{N}(\theta;z,\sigma_\theta^2)1_{\mathcal{B}}(\theta) \approx \mathcal{N}(\theta;z,\sigma_\theta^2) \tag{13}$$

which can be justified in practice by assuming that the standard deviation of the measurement noise σ_θ is small compared to the surveillance region $\mathcal{B}$. This leads to the following approximation for the integral

$$\int P_{D,k}(\xi)g(z|\xi)v_{k|k-1}(\xi)d\xi \approx P_{D,k}\sum_{i=1}^{J_{k-1}} w_{k|k-1}^{(i)}q_k^{(i)}(z) + \frac{w_k^b}{V_{\mathcal{B}}} \tag{14}$$

Substituting (13) and (14) into (10) and (11) gives us the following approximation for the posterior PHD, which is similar to the update equations given in [9]:

$$v_k(\theta,\phi,0) \approx \sum_{i=1}^{J_{k-1}} w_{m,k}^{(i)}\mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)},\Omega_{k|k-1}^{(i)}) + \sum_{z\in Z}\sum_{i=1}^{J_{k-1}} w_{s,k}^{(i)}\mathcal{N}(\theta,\phi;(\theta,\phi)_k^{(i)},\Omega_k^{(i)}) \tag{15}$$

$$v_k(\theta,\phi,1) \approx \sum_{z\in Z} w_{b,k}^{(i)}\mathcal{N}(\theta;z,\sigma_\theta^2)\mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2) \tag{16}$$

where

$$w_{m,k}^{(i)} = (1-P_{D,k})w_{k|k-1}^{(i)} \tag{17}$$

$$w_{s,k}^{(i)} = \frac{P_{D,k}q_k^{(i)}(z)w_{k|k-1}^{(i)}}{\kappa_k(z) + P_{D,k}\sum_{i=1}^{J_{k-1}} w_{k|k-1}^{(i)}q_k^{(i)}(z) + \frac{w_k^b}{V_{\mathcal{B}}}} \tag{18}$$

$$w_{b,k}^{(i)} = \frac{w_k^b/V_{\mathcal{B}}}{\kappa_k(z) + P_{D,k}\sum_{i=1}^{J_{k-1}} w_{k|k-1}^{(i)}q_k^{(i)}(z) + \frac{w_k^b}{V_{\mathcal{B}}}} \tag{19}$$

Since newborn targets at time k become surviving targets at time $k+1$, the predicted PHD for surviving targets at time $k+1$ is obtained by propagating the sum of both the $\beta=0$ and $\beta=1$ components of the posterior PHD at time k.

III. GMCPHD Filter with Partially Uniform Target Birth Model (PUB-GMCPHD)

The CPHD filter is a generalisation of the PHD filter which relaxes the assumption that the number of targets follows a Poisson distribution. To achieve this, the filter propagates the discrete probability distribution on the number of targets (or cardinality distribution), thereby allowing it to take on an arbitrary form. In this section, we derive an approximation of the GMCPHD filter using the PUB model, which we refer to as the PUB-GMCPHD filter.

A. Prediction

Let the posterior PHD at time $k-1$ be a Gaussian mixture of the same form as in (2). Let ρ_{k-1} denote the posterior cardinality distribution at time $k-1$, and $\rho_{\Gamma,k}$ the prior cardinality distribution of spontaneous births at time k. Then, as per the standard GMCPHD filter, the predicted cardinality distribution at time k is given by the convolution [7]

$$\rho_{k|k-1}(n) = \sum_{j=0}^{n}\rho_{\Gamma,k}(n-j)\sum_{l=j}^{\infty} C_j^l\rho_{k-1}(l)P_{S,k}^j(1-p_{S,k})^{l-j} \tag{20}$$

where $C_j^l = \frac{l!}{j!(l-j)!}$ is the binomial coefficient. The PHD of target births is the same as defined in (4), and the predicted PHD at time $k-1$ is the same as (5). As we did for the PHD filter, we again augment the state with the variable β to distinguish surviving components from the birth component, leading to the definition (7) for the predicted PHD.

B. Update

Using the definition in (8) for the detection probability, and making the following definitions to simplify the notation,

$$\chi = \frac{\langle \Upsilon_k^1[v_{k|k-1}, Z_k], \rho_{k|k-1}\rangle}{\langle \Upsilon_k^0[v_{k|k-1}, Z_k], \rho_{k|k-1}\rangle} \tag{21}$$

$$\chi(z) = \frac{\langle \Upsilon_k^1[v_{k|k-1}, Z_k - \{z\}], \rho_{k|k-1}\rangle}{\langle \Upsilon_k^0[v_{k|k-1}, Z_k], \rho_{k|k-1}\rangle} \tag{22}$$

where $\langle \cdot, \cdot \rangle$ is the inner product between two functions, the CPHD update can be expressed as

$$\rho_k(n) = \frac{\Upsilon_k^0[v_{k|k-1}, Z_k](n)}{\langle \Upsilon_k^0[v_{k|k-1}, Z_k], \rho_{k|k-1}\rangle}\rho_{k|k-1}(n) \tag{23}$$

$$v_k(\theta,\phi,0) = (1-P_{D,k})\chi v_{k|k-1}(\theta,\phi,0) + \sum_{z\in Z}\psi_{k,z}(\theta,\phi,0)\chi(z)v_{k|k-1}(\theta,\phi,0) \tag{24}$$

$$v_k(\theta,\phi,1) = \sum_{z\in Z}\psi_{k,z}(\theta,\phi,1)\chi(z)v_{k|k-1}(\theta,\phi,1) \tag{25}$$

where

$$\Upsilon_k^u[v,Z](n) = \sum_{j=0}^{\min(|Z|,n)} (|Z|-j)!P_{K,k}(|Z|-j)P_{j+u}^n \frac{\langle 1-P_{D,k}, v\rangle^{n-(j+u)}}{\langle 1, v\rangle^n} e_j(\Xi_k(v,Z)) \tag{26}$$

$$\psi_{k,z}(\theta,\phi,\beta) = \frac{\langle 1, \kappa_k\rangle}{\kappa_k(z)} g(z|\theta,\phi)P_{D,k}(\theta,\phi,\beta) \tag{27}$$

$$\Xi_k(v,Z) = \{\langle v, \psi_{k,z}\rangle : z\in Z\} \tag{28}$$

$$P_{j+u}^n = \frac{n!}{(n-(j+u))!} \tag{29}$$

$P_{K,k}(\cdot)$ is the cardinality distribution of clutter at time k, and $e_j(\cdot)$ is the elementary symmetric function of order j. The inner product between $v_{k|k-1}$ and $\psi_{k,z}$ is given by the following,

$$\langle v_{k|k-1}, \psi_{k,z}\rangle = \frac{\langle 1, \kappa_k\rangle}{\kappa_k(z)}\left(\frac{w_k^b}{V_{\mathcal{B}}} + P_{D,k}\sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} q_k^{(i)}(z)\right) \tag{30}$$

Hence, the elementary symmetric functions are calculated over the set defined by

$$\Xi_k(v_{k|k-1}, Z) = \left\{\frac{\langle 1, \kappa_k\rangle}{\kappa_k(z)}\left(\frac{w_k^b}{V_{\mathcal{B}}} + P_{D,k}\sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} q_k^{(i)}(z)\right) : z\in Z\right\}$$

The remaining inner products required in the CPHD calculations are

$$\langle 1, v_{k|k-1} \rangle = \langle 1, w_{k|k-1} \rangle + w_k^b \tag{31}$$

$$\langle 1 - P_{D,k}, v_{k|k-1} \rangle = (1 - P_{D,k}) \langle 1, w_{k|k-1} \rangle \tag{32}$$

which results in the following equation for Υ_k^u

$$\Upsilon_k^u[v_{k|k-1}, Z](n) = \sum_{j=0}^{\min(|Z|,n)} (|Z|-j)! P_{K,k}(|Z|-j)(1-P_D)^{n-(j+u)} \frac{\langle 1, w_{k|k-1} \rangle^{n-(j+u)}}{\left(\langle 1, w_{k|k-1} \rangle + w_k^b\right)^n} P_{j+u}^n e_j \left(\Xi_k(v_{k|k-1}, Z)\right) \tag{33}$$

Finally, the posterior PHD is approximated by

$$v_k(\theta, \phi, 0) \approx \sum_{i=1}^{J_{k|k-1}} w_{k,m}^{(i)} \mathcal{N}(\theta, \phi; (\theta, \phi)_{k|k-1}^{(i)}, \Omega_{k|k-1}^{(i)}) + \sum_{z \in Z} \sum_{i=1}^{J_{k|k-1}} w_{k,s}^{(i)}(z) \mathcal{N}(\theta, \phi; (\theta, \phi)_k^{(i)}, \Omega_k^{(i)}) \tag{34}$$

$$v_k(\theta, \phi, 1) \approx \sum_{z \in Z} w_{k,b}(z) \mathcal{N}(\theta; z, \sigma_\theta^2) \mathcal{N}(\phi; \bar{\phi}, \sigma_\phi^2) \tag{35}$$

where

$$w_{k,m}^{(i)} = (1 - P_{D,k}) w_{k|k-1}^{(i)} \chi \tag{36}$$

$$w_{k,s}^{(i)}(z) = \frac{\langle 1, \kappa_k \rangle}{\kappa_k(z)} P_{D,k} w_{k|k-1}^{(i)} q_k^{(i)}(z) \chi(z) \tag{37}$$

$$w_{k,b}(z) = \frac{\langle 1, \kappa_k \rangle}{\kappa_k(z)} \frac{w_k^b}{V_{\mathcal{B}}} \chi(z) \tag{38}$$

which is a Gaussian mixture as required. As we did for the PHD filter, the $\beta = 0$ and $\beta = 1$ components of the posterior PHD at time k are added together in the prediction at time $k+1$. The derivations of equations (30), (31), (34), and (35) are given in the appendix.

The calculation of the posterior cardinality distribution remains unchanged from the conventional GMCPHD filter, except for the new definition of $\Upsilon_k^u[v_{k|k-1}, Z](n)$. Hence, the posterior cardinality distribution is given by (23).

IV. Simulation Results

In this section, we present simulation results demonstrating the performance of the PUB-GMPHD/CPHD filters applied to single-sensor bearings-only multi-target filtering. This is an important problem, especially in the context of covert surveillance, because bearing measurements often constitute the only reliable output from sensors which operate in passive mode. It is challenging due to the non-linear measurement model, and the potential for poor observability of targets. The effect of observability in bearings-only target motion analysis has been studied in [14–17], but for the case of bearings-only multi-target tracking in clutter, the effects are difficult to characterise in a theoretical context. Despite it's practical importance, numerical results on this problem are scarcely reported in the literature.

To analyse the performance of the PUB-GMPHD/CPHD filters, we compare them with the corresponding conventional filters under Gaussian mixture birth models with various numbers of components. We simulate a scenario consisting of a single bearings-only sensor on board a manoeuvering platform. The position of this platform is known at all times and denoted in Cartesian coordinates as (x_k^s, y_k^s) for time index k. There are up to six targets in the sensor's detection region, each following a nearly constant velocity motion model, defined as follows. At time k, let target t have Cartesian position coordinates (x_k^t, y_k^t) and instantaneous velocity vector $(\dot{x}_k^t, \dot{y}_k^t)$, such that the target's state vector can be defined as

$$\mathbf{x}_{k,t} = \left[x_k^t, y_k^t, \dot{x}_k^t, \dot{y}_k^t\right]^T .$$

For notational convenience, we also define a similar vector $\mathbf{x}_{k,s}$ for the known state of the sensor platform. The target state evolves according to the discrete-time transition

$$\mathbf{x}_{k+1,t} = \mathbf{F}\mathbf{x}_{k,t} + \mathbf{\Gamma}\mathbf{v}_k \tag{39}$$

$$\mathbf{F} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \otimes \boldsymbol{I}_2, \quad \mathbf{\Gamma} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \otimes \boldsymbol{I}_2, \tag{40}$$

where T is the time between measurements, and $\mathbf{v}_k \sim \mathcal{N}(0, \boldsymbol{Q})$ is a 2×1 i.i.d Gaussian process noise vector with $\boldsymbol{Q} = \sigma_v^2 \boldsymbol{I}_2$ where $\sigma_v = 0.005$ m/s^2 is the standard deviation of the process noise acceleration.

The sensor measures target t bearing according to the model

$$z_k^t = h(\mathbf{x}_{k,t}, \mathbf{x}_{k,s}) + w_k \tag{41}$$

$$h(\mathbf{x}_{k,t}, \mathbf{x}_{k,s}) = \arctan\left(\frac{x_k^t - x_k^s}{y_k^t - y_k^s}\right) \tag{42}$$

where $w_k \sim \mathcal{N}(0, \sigma_\theta^2)$ is zero-mean white Gaussian measurement noise with standard deviation $\sigma_\theta = 1^\circ$. The sensor generates one set of measurements every 10 seconds, and the detection probability is 0.95 for all targets. The measurement sets also contain false detections distributed uniformly across the bearing space, the number of which follows a Poisson distribution with a known mean of 25 per scan. Three targets are present at the beginning of the scenario, with two arriving at time 300 s, and one more at time 600 s. One target leaves at 2200 s, and another leaves at 2600 s. The scenario geometry is shown in Figure 1, and an example of the measurements processed by the filter is shown in Figure 2.

To handle the non-linear measurement model, all filters use local linearization based on the extended Kalman filter (EKF). Clearly, other types of non-linear filtering algorithms may be used in place of the EKF, such as the Unscented Kalman filter [18], Cubature Kalman filter [19], or Shifted Rayleigh filter [20], but comparing the performance of these algorithms is beyond the scope of this paper. In all simulations, the expected number of new targets per scan is set to 0.05, the Malahanobis distance threshold for merging components in the Gaussian mixture is set to 2, and the weight threshold for elimination of mixture components is 10^{-5}. All performance metrics shown are averaged over 500 Monte Carlo runs, with fixed ground truth and independent measurement realisations.

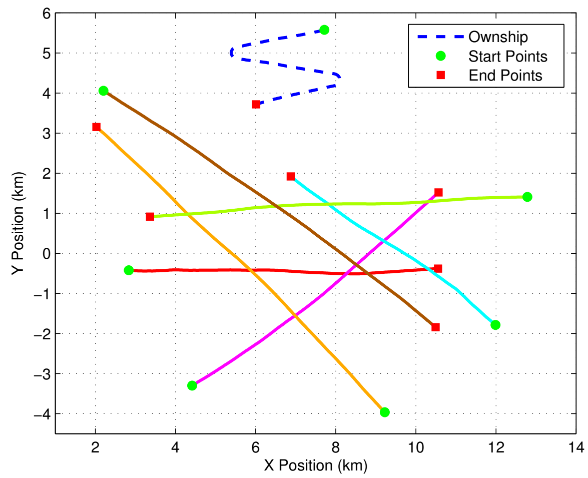


Fig. 1: Target-Observer geometry

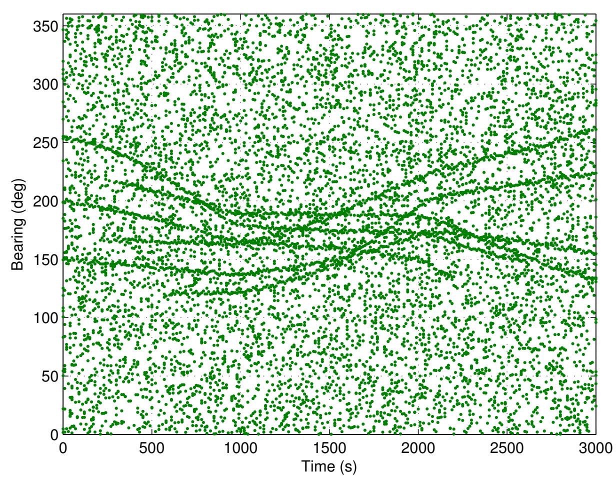


Fig. 2: Example bearing measurements

The results of both PUB-GMPHD and PUB-GMCPHD are compared against the corresponding filters with Gaussian mixture birth models with different numbers of components. To construct the GM birth models, we position the means of the Gaussians evenly across the bearing space, at a nominal range of 12 km from the sensor with a range standard deviation of 4 km. The bearing standard deviation is set such that the gaps between the means are adequately covered. For this analysis, we have tested the filters with various GM birth models, where the number of Gaussians increases by factors of 4, with a minimum of 4 and a maximum of 1024 components. We refer to these models as GMX, where X is the number of Gaussians. For the GM4 model, we use a bearing standard deviation of $40°$, for GM16 we use $10°$, for GM64 we use $2.5°$, and for GM256 and GM1024 we use $1°$.

Figure 3 illustrates two of the GM birth models tested, and Figure 4 shows the uniform birth model.

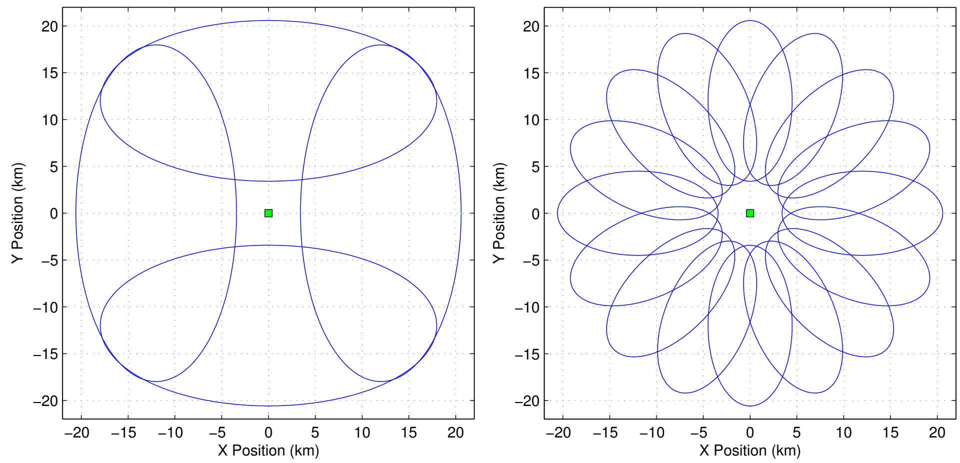


Fig. 3: GM4 and GM16 birth models. Each ellipse represents the 90% confidence region of a Gaussian component, and the square marker is the sensor location.

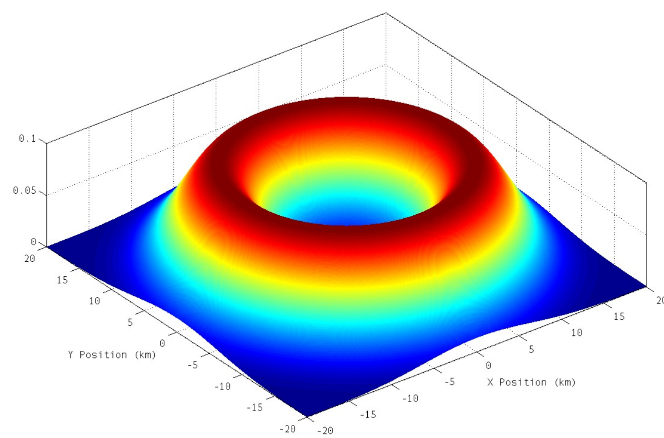


Fig. 4: Birth model with uniform bearing distribution

Figure 5 shows the Optimal Sub-Pattern Assignment (OSPA) distance [12] (with order 2 and cutoff 4 km) for the PUB-GMPHD/CPHD filters. As expected, we find that CPHD outperforms PHD, which is due to the reduced variability in the cardinality estimates. Figure 6 shows the average increase in the OSPA for each GM birth model compared to the PUB model, expressed as a percentage of the OSPA for the PUB model. This is computed at all times k using the formula

$$\Delta_{OSPA}(k) = \frac{100}{N_{MC}} \sum_{i=1}^{N_{MC}} \frac{O^{(i)}_{\text{GM}X}(k) - O^{(i)}_{PUB}(k)}{O^{(i)}_{PUB}(k)} \tag{43}$$

where N_{MC} is the number of Monte Carlo runs, $O^{(i)}_{\text{GM}X}(k)$ is the OSPA at time k on run i using the GMX birth model, and $O^{(i)}_{PUB}(k)$ is the OSPA at time k on run i using the PUB model.

Table I summarises the results by showing the time averaged performance under each birth model. The first column ($\bar{\Delta}_{OSPA}$) shows the time averaged percentage increase in OSPA under the GM birth model compared to the PUB model, calculated by averaging (43) over all k. The second column (Exec Tme) shows the average execution time, expressed as a multiple of the execution time under the PUB model.

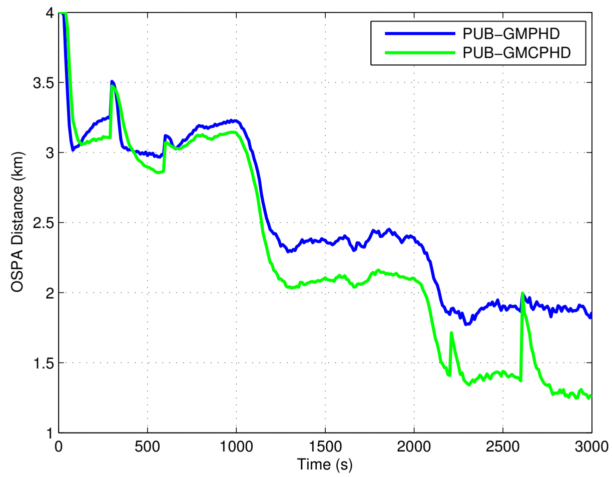
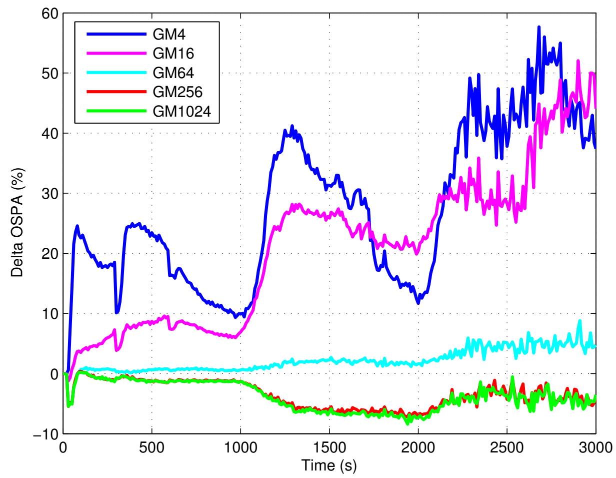


Fig. 5: OSPA for PUB-GMPHD/CPHD

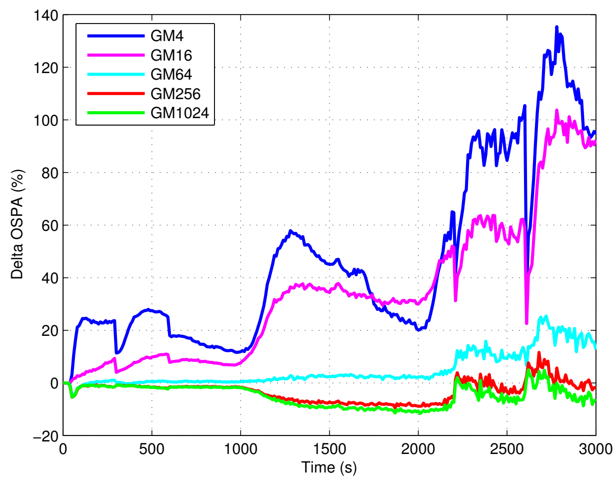
TABLE I: Filter performance with different birth models

Birth Model	GMPHD		GMCPHD	
	$\bar{\Delta}_{OSPA}$ (%)	Exec Tme	$\bar{\Delta}_{OSPA}$ (%)	Exec Tme
PUB	0	1	0	1
GM4	27.53	3.41	46.70	2.82
GM16	21.03	1.02	34.14	1.00
GM64	2.09	1.39	4.79	1.26
GM256	-3.51	2.40	-3.07	2.15
GM1024	-3.77	6.20	-4.80	4.20

From Figure 6 and Table I it is clear that using the GM4 model to represent the PHD of new targets has led to higher estimation error and execution time than the filter with uniform birth. The higher error is mainly due to the worse localization of target states induced by the poor initialization afforded by the birth model. In non-linear filtering, it is well known that the initial conditions can have a significant and lasting effect on error performance,



(a) GMPHD



(b) GMCPHD

Fig. 6: Average percentage increase in OSPA for GM birth models compared to the PUB model

which is manifested here as an increase in OSPA distance. The higher execution time is due to the fact that the filter retains the diffuse Gaussians from the birth PHD for longer before they are eliminated from the mixture, thereby increasing the average number of components in the posterior. This effect occurs in bearings-only tracking because when the bearing component of the initial covariance is very large, it takes several update cycles to reduce to a value closer to the variance of the measured bearings. This delay in convergence means that there is more opportunity for false measurements to confirm the presence of a target, hence it takes longer for false components to be discarded.

As we increase the number of Gaussians in the birth model, the average OSPA improves, as can be seen from Figure 6 and Table I. The GM64 model performs slightly worse than the uniform model, and GM256 performs slightly better, however, a further increase to 1024 components provides very marginal improvement, with a significant increase in computational cost. Note that the differences in Δ_{OSPA} percentages between the PHD and CPHD filters are due to the difference in absolute error between the filters. For the uniform birth model, the PHD filter has higher error than the CPHD (see Figure 5), which leads to lower percentage differences than for the CPHD filter.

From these results we can conclude that using the uniform target birth model avoids the potential computational overhead that can occur when using a Gaussian mixture birth model which is either too large or too small. The uniform birth model also provides error performance which is similar to using a large GM birth model, with some small penalty associated with approximating the posterior as a Gaussian mixture.

V. Conclusion

In this paper we have derived new forms of the GMPHD/CPHD filters which allow a uniform distribution to be used in the prior target birth model for the part of the state space which is directly observed by the measurements. Although this leads to a predicted PHD which is not a Gaussian mixture, if the measurements are Gaussian, it is possible to obtain an accurate Gaussian mixture approximation to the posterior PHD after the update. We tested the resulting filters by performing Monte Carlo simulations on a non-linear bearings-only tracking scenario. The advantage of the partially uniform approach is that it provides similar performance to a GM birth model with a large number of Gaussians, while avoiding the need to choose the number and locations of the Gaussian birth components. Furthermore, the proposed method avoids any computational overhead that may result from poor choices for the Gaussian mixture parameters.

Appendix

Derivation of equation (12):

$$\begin{aligned}
\int P_{D,k}(\xi)g(z|\xi)v_{k|k-1}(\xi)d\xi &= \int\int\int P_{D,k}(\theta,\phi,\beta)g(z|\theta,\phi,\beta)v_{k|k-1}(\theta,\phi,\beta)d\theta d\phi d\beta \\
&= \int\int\sum_{\beta=0}^{1} P_{D,k}(\theta,\phi,\beta)g(z|\theta,\phi,\beta)v_{k|k-1}(\theta,\phi,\beta)d\theta d\phi \\
&= \int\int P_{D,k}\sum_{i=1}^{J_{k-1}} w_{k|k-1}^{(i)}\mathcal{N}(z;\theta_{k|k-1}^{(i)},\sigma_\theta^2)\mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)},\Omega_{k|k-1}^{(i)})d\theta d\phi \\
&\quad + w_k^b\int\int\mathcal{N}(\theta;z,\sigma_\theta^2)\mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2)U(\theta;\mathcal{B})d\theta d\phi \\
&= \sum_{i=1}^{J_{k-1}} P_{D,k}w_{k|k-1}^{(i)}q_k^{(i)}(z)\int\int\mathcal{N}(\theta,\phi;(\theta,\phi)_k^{(i)},\Omega_k^{(i)})d\theta d\phi \\
&\quad + w_k^b\int\mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2)d\phi\int\mathcal{N}(\theta;z,\sigma_\theta^2)\frac{1_{\mathcal{B}}(\theta)}{V_{\mathcal{B}}}d\theta
\end{aligned}$$

Derivation of equation (30):

$$
\begin{aligned}
\langle v_{k|k-1}, \psi_{k,z} \rangle &= \int\int\int v_{k|k-1}(\theta,\phi,\beta)\psi_{k,z}(\theta,\phi,\beta)d\theta d\phi d\beta \\
&= \int\int \sum_{\beta=0}^{1} v_{k|k-1}(\theta,\phi,\beta)\psi_{k,z}(\theta,\phi,\beta)d\theta d\phi \\
&= \int\int \left[\sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)}, \Omega_{k|k-1}^{(i)}) \frac{\langle 1,\kappa_k \rangle}{\kappa_k(z)} \mathcal{N}(z;\theta,\sigma_\theta^2) P_{D,k}(r,\theta,0) \right] \\
&\quad + w_k^b \mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2) U(\theta;\mathcal{B}) \frac{\langle 1,\kappa_k \rangle}{\kappa_k(z)} \mathcal{N}(z;\theta,\sigma_\theta^2) P_{D,k}(\theta,\phi,1) d\theta d\phi \\
&= \frac{\langle 1,\kappa_k \rangle}{\kappa_k(z)} P_{D,k} \sum_{i=1}^{J_{k|k-1}} \int\int w_{k|k-1}^{(i)} q_k^{(i)}(z) \mathcal{N}(\theta,\phi;(\theta,\phi)_k^{(i)}, \Omega_k^{(i)}) d\phi d\theta \\
&\quad + \frac{\langle 1,\kappa_k \rangle}{\kappa_k(z)} w_k^b \int \mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2) d\phi \int \mathcal{N}(\theta;z,\sigma_\theta^2) \frac{1_{\mathcal{B}}(\theta)}{V_{\mathcal{B}}} d\theta \\
&= \frac{\langle 1,\kappa_k \rangle}{\kappa_k(z)} \left(\Psi w_k^b + P_{D,k} \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} q_k^{(i)}(z) \right)
\end{aligned}
$$

Derivation of equation (31):

$$
\begin{aligned}
\langle 1, v_{k|k-1} \rangle &= \int\int \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)}, \Omega_{k|k-1}^{(i)}) + w_k^b \mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2) U(\theta;\mathcal{B}) d\theta d\phi \\
&= \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} + w_k^b \int \mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2) d\phi \int U(\theta;\mathcal{B}) d\theta \\
&= \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} + w_k^b \\
&= \langle 1, w_{k|k-1} \rangle + w_k^b
\end{aligned}
$$

Derivation of equation (34):

$$
\begin{aligned}
v_k(\theta,\phi,0) &= (1-P_{D,k})\chi v_{k|k-1}(\theta,\phi,0) + \sum_{z\in Z} \psi_{k,z}(\theta,\phi,0)\chi(z) v_{k|k-1}(\theta,\phi,0) \\
&= (1-P_{D,k})\chi \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)}, \Omega_{k|k-1}^{(i)}) \\
&\quad + \sum_{z\in Z} \sum_{i=1}^{J_{k|k-1}} P_{D,k} \frac{\langle 1,\kappa_k \rangle}{\kappa_k(z)} \chi(z) w_{k|k-1}^{(i)} \mathcal{N}(\theta;z,\sigma_\theta^2) \mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)}, \Omega_{k|k-1}^{(i)}) \\
&= (1-P_{D,k})\chi \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\theta,\phi;(\theta,\phi)_{k|k-1}^{(i)}, \Omega_{k|k-1}^{(i)}) \\
&\quad + \sum_{z\in Z} \sum_{i=1}^{J_{k|k-1}} P_{D,k} \frac{\langle 1,\kappa_k \rangle}{\kappa_k(z)} \chi(z) w_{k|k-1}^{(i)} q_k^{(i)}(z) \mathcal{N}(\theta,\phi;(\theta,\phi)_k^{(i)}, \Omega_k^{(i)})
\end{aligned}
$$

Derivation of equation (35):

$$\begin{aligned}
v_k(\theta,\phi,1) &= \sum_{z\in Z} \psi_{k,z}(\theta,\phi,1)\chi(z)v_{k|k-1}(\theta,\phi,1) \\
&= \sum_{z\in Z} \chi(z)\frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)}\mathcal{N}(\theta;z,\sigma_\theta^2)v_{k|k-1}(\theta,\phi,1) \\
&= \sum_{z\in Z} \chi(z)\frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)}w_k^b\mathcal{N}(\theta;z,\sigma_\theta^2)\mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2)U(\theta;\mathcal{B}) \\
&= \sum_{z\in Z} \chi(z)\frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)}w_k^b\mathcal{N}(\theta;z,\sigma_\theta^2)\mathcal{N}(\phi;\bar{\phi},\sigma_\phi^2)\frac{1}{V_{\mathcal{B}}}
\end{aligned}$$

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