On the stability and the approximation of branching distribution flows, with applications to nonlinear multiple target filtering

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Abstract

We analyse the exponential stability properties of a class of measure-valued equations arising in nonlinear multi-target filtering problems. We also prove the uniform convergence properties of a rather general class of stochastic filtering algorithms, including sequential Monte Carlo type models and mean field particle interpretation models. We illustrate these results in the context of the Bernoulli and the Probability Hypothesis Density filter, yielding what seems to be the first results of this kind in this subject.

Keywords: Measure-valued equations, nonlinear multi-target filtering, Bernoulli filter, Probability hypothesis density filter, interacting particle systems, particle filters, sequential Monte Carlo methods, exponential concentration inequalities, semigroup stability, functional contraction inequalities.

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1 Introduction

Let $(E_n)_{n\geq 0}$ be a sequence of measurable spaces equipped with the σ-fields $(\mathcal{E}_n)_{n\geq 0}$, and for each with $n \geq 0$, denote $\mathcal{M}(E_n)$, $\mathcal{M}_+(E_n)$ and $\mathcal{P}(E_n)$ the set of all finite signed measures, the subset of positive measures and the subset of probability measures, respectively, over the space E_n. The aim of this work is to present a stochastic interacting particle interpretation for numerical solutions of the general measure-valued dynamical systems $\gamma_n \in \mathcal{M}_+(E_n)$ defined by the following non-linear equation

$$\gamma_n(dx_n) = \left(\gamma_{n-1}Q_{n,\gamma_{n-1}}\right)(dx_n) := \int_{E_{n-1}} \gamma_{n-1}(dx_{n-1})Q_{n,\gamma_{n-1}}(x_{n-1}, dx_n) \tag{1.1}$$

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with initial measure $\gamma_0 \in \mathcal{M}_+(E_0)$, and positive and bounded integral operators $Q_{n,\gamma}$ from E_{n-1} into E_n, indexed by the time parameter $n \geq 1$ and the set of measures $\gamma \in \mathcal{M}_+(E_n)$.

This class of measure-valued equations arises in a natural way in the analysis of the first moments evolution of nonlinear branching processes, as well as in signal processing and more particularly in multiple targets tracking models. A pair of filtering models is discussed in some details in section 1.1.1 and in section 1.1.2. In the context of multiple targets tracking problems these measure-valued equations represents the first-order statistical moments of the conditional distributions of the target occupation measures given observation random measures obscured by clutter, detection uncertainty and data association uncertainty.

As in most of the filtering problems encountered in practice, the initial distribution of the targets is usually unknown. It is therefore essential to check wether or not the filtering equation "forgets" any erroneous initial distribution. For a thorough discussion on the stability properties of traditional nonlinear filtering problems with a detailed overview of theoretical developments on this subject, we refer to the book [5] and to the more recent article by M. L. Kleptsyna and A. Y. Veretennikov [12]. Besides the fact that significant progress has been made in the recent years in the rigorous derivation of multiple target tracking nonlinear equations (see for instance [4, 14, 24, 19]), up to our knowledge the stability and the robustness properties of these measure-valued models have never been addressed so far in the literature on the subject. One aim of this paper is to study one such important property: the exponential stability properties of multiple target filtering models. We present an original and general perturbation type technique combining the continuity property and the stability analysis of nonlinear semigroups of the form (1.1). A more thorough presentation of these results is provided in section 1.2 dedicated to the statement of the main results of the present article. The detailed presentation of this perturbation technique can be found in section 3.

On the other hand, while the integral equation (1.1) appears to be simple at first glance, numerical solutions are computationally intensive, often requiring integrations in high dimensional spaces. One natural way to solve the non-linear integral equation (1.1) is to use find a judicious probabilistic interpretation of the normalized distributions flow given below

$$\eta_n(dx_n) := \gamma_n(dx_n)/\gamma_n(1)$$

To describe with some conciseness these stochastic models, it is important to observe that the pair process $(\gamma_n(1), \eta_n) \in (\mathbb{R}_+ \times \mathcal{P}(E_n))$ satisfies an evolution equation of the following form

$$(\gamma_n(1), \eta_n) = \Gamma_n(\gamma_{n-1}(1), \eta_{n-1}) \tag{1.2}$$

Let the mappings $\Gamma_n^1 : \mathbb{R}_+ \times \mathcal{P}(E_n) \to \mathbb{R}_+$ and $\Gamma_n^2 : \mathbb{R}_+ \times \mathcal{P}(E_n) \to \mathcal{P}(E_n)$, denote the first and the second components of Γ_n respectively. By construction, we notice that the total mass process can be computed using the recursive formula

$$\gamma_{n+1}(1) = \gamma_n(G_{n,\gamma_n}) = \eta_n(G_{n,\gamma_n})\ \gamma_n(1) \quad \text{with} \quad G_{n,\gamma_n} := Q_{n+1,\gamma_n}(1) \tag{1.3}$$

Suppose that we are given an approximation $\left(\gamma_n^N(1), \eta_n^N\right)$ of the pair $(\gamma_n(1), \eta_n)$ at some time horizon n, where N stands for some precision parameter; that is $\left(\gamma_n^N(1), \eta_n^N\right)$ converges (in some sense) to $(\gamma_n(1), \eta_n)$, as $N \to \infty$. Then, the N-approximation of the measure γ_n is given by $\gamma_n^N = \gamma_n^N(1) \times \eta_n^N$. The central idea behind any approximation model is to ensure that the total mass process at time $(n+1)$ defined by

$$\gamma_{n+1}^N(1) = \eta_n^N(G_{n,\gamma_n^N})\ \gamma_n^N(1) \tag{1.4}$$

can be "easily" computed in terms of the N-approximation measures γ_n^N. Assuming that the initial mass $\gamma_0(1) = \gamma_0^N(1)$ is known, the next step is to find some strategy to approximate the quantities $\Gamma_{n+1}^2(\gamma_n^N(1), \eta_n^N)$ by some N-approximation measures η_{n+1}^N, and to set $\gamma_{n+1}^N = \gamma_{n+1}^N(1) \times \eta_{n+1}^N$.

The local fluctuations of η_n^N around the measures $\Gamma_n^2(\gamma_{n-1}^N(1), \eta_{n-1}^N)$ is defined in terms of a collection of random fields W_n^N :

$$W_n^N := \sqrt{N}\ \left[\eta_n^N - \Gamma_n^2(\gamma_{n-1}^N(1), \eta_{n-1}^N)\right] \Longleftrightarrow \eta_n^N = \Gamma_n^2\left(\gamma_{n-1}^N(1), \eta_{n-1}^N\right) + \frac{1}{\sqrt{N}}\ W_n^N \tag{1.5}$$

which satisfies for any $r \geq 1$ and any test function f with uniform norm $\|f\| \leq 1$,

$$\mathbb{E}\left(W_n^N(f) \mid \mathcal{F}_{n-1}^N\right) = 0 \quad \text{and} \quad \mathbb{E}\left(\left|W_n^N(f)\right|^r \mid \mathcal{F}_{n-1}^N\right)^{\frac{1}{r}} \leq a_r \tag{1.6}$$

where $\mathcal{F}_{n-1}^N = \sigma\left(\eta_p^N, 0 \leq p < n\right)$ is the σ-field generated by the random measures η_p^N, $0 \leq p < n$, while b and a_r are universal constants whose values do not depend on the precision parameter N. The stochastic analysis of the resulting particle approximation model relies on the analysis of the propagation of the local sampling errors defined in (1.5). The main objective is to control, at any time horizon n, the fluctuations of the random measures (γ_n^N, η_n^N) around their limiting values (γ_n, η_n) defined by the following random fields:

$$V_n^{\gamma,N} := \sqrt{N}\ \left[\gamma_n^N - \gamma_n\right] \quad \text{with} \quad V_n^{\eta,N} := \sqrt{N}\ \left[\eta_n^N - \eta_n\right]. \tag{1.7}$$

The construction of the N-approximation measures η_n^N is far from being unique. In the present article, we devise three different classes of stochastic particle approximation models. These stochastic algorithms are discussed in section 4. The first one is a mean field particle interpretation of the flow of probability measures η_n, and it is presented in section 4.1. The second model is an interacting particle association model while the third one is a combination of these two approximation algorithms. These pair of approximation models are respectively discussed in section 4.2 and in section 4.3. In the context of multi-target tracking models, the first two approximation models are closely related to the the sequential Monte Carlo technique presented in the series of articles [20, 21, 25, 26, 27, 28], and respectively, the Gaussian mixture Probability Hypothesis Density filter discussed in the article by B.-N. Vo, and W.-K. Ma [22, 23], and the the Rao-Blackwellized Particle multi-target filters presented by S. Sarkka, A. Vehtari, and J. Lampinen in [17, 18]. These modern stochastic algorithms are rather simple to implement and computationally tractable, and they exhibit excellent performance.

Nevertheless, despite advances in recent years [3, 11, 21], these Monte Carlo particle type multi-target filters remain poorly understood theoretically. One aim of this article is to present a novel class of stochastic algorithms with a refined analysis including uniform convergence results w.r.t. the time parameter. We also illustrate these results in the context of multi-target tracking models, yielding what seems to be the first uniform results of this type in this subject.

The rest of the article is organized as follows: In section 1.1 we illustrate the abstract measure-valued equations (1.1) with two recent multi-target filters models, namely the Bernoulli filter and the Probability Hypothesis Density filter (*abbreviate PHD filter*). Section 1.2 is devoted to the statement of our main results. In section 2, we describe the semigroups and the continuity properties of the nonlinear equation 1.1. We show that this semigroup analysis can be applied to analyse the convergence of the Bernoulli and the PHD approximation filters. Section 3 is devoted to the stability properties of nonlinear measure-valued processes of the form (1.2). We present a perturbation technique and a series of functional contraction inequalities. In the next three sections, we illustrate these results in the context of Feynman-Kac models, as well as Bernoulli and PHD models. Section 4 is concerned with the detailed presentation and the convergence analysis of three different classes of particle type approximation models, including mean field type particle approximations and particle association stochastic algorithms. Finally, the appendix of the article contains most of technical proofs in the text.

1.1 Measure-valued systems in Multi-target tracking

The measure-valued process given by (1.1) is a generalisation of Feynman-Kac measures. Its continuous time version naturally arise in the modeling and analysis of the first moments of spatial branching process [5, 8].

Our major motivation for studying this class of measure-valued system stems from advanced signal processing, more specifically, multiple target tracking. Driven primarily in the early 1970's by aerospace applications such as radar, sonar, guidance, navigation, and air traffic control, today multi-target filtering has found applications in many diverse disciplines, see for example the texts [1], [2] [15] and references therein. These nonlinear filtering problems deal with jointly estimating the number and states of several interacting targets given a sequence of partial observations corrupted by noise, false measurements as well as miss-detection. This rapidly developing subject is, arguably, one of the most interesting contact points between the theory of spatial branching processes, mean field particle systems and advanced signal processing.

The first connections between stochastic branching processes and multi-target tracking seem to go back to the article by S. Mori, et. al. [16] published in 1986. However it was Mahler's systematic treatment of multi-sensor multi-target filtering using random finite sets theory [10, 9, 13, 14] that lead to the development novel multi-target filters and sparked world wide interests. To motivate the article, we briefly outline two recent multi-target filters that do not fit the standard Feynman-Kacs framework, but fall under the umbrella of the measure-valued equation (1.1). The first is the Bernoulli filter for joint detection and tracking of a single target while the second is the Probability Hypothesis Density filter.

1.1.1 Bernoulli filtering

A basic problem in target tracking is that the target of interest may not always be present and exact knowledge of target existence/presence cannot be determined from observations due to clutter and detection uncertainty [15]. The *Bernoulli* filter is a generalisation of the standard Bayes filter, which accommodates presence and absence of the target [24]. In a Bernoulli model, the birth of the target at time $n+1$ is modelled by a measure μ_{n+1}

on E_{n+1}. The target enters the scene with a probability $\mu_{n+1}(1) < 1$ and its state is distributed according to the normalised measure $\mu_{n+1}/\mu_{n+1}(1)$. At time n, a target X_n has a probability $s_n(X_n)$ of surviving to the next time and evolve to a new state according to a given elementary Markov transition M_{n+1} from E_n into E_{n+1}. At time $n+1$, the target (if it exists) generates with probability $d_{n+1}(X_{n+1})$ an observation Y_{n+1} on some auxiliary state space, say E^Y_{n+1} with likelihood function $l_{n+1}(X_{n+1}, y)$. This so-called Bernoulli observation point process is superimposed with an additional and independent Poisson point process with intensity function $h_n > 0$ to form the occupation (or counting) measure observation process $\mathcal{Y}_{n+1} = \sum_{1\leq i\leq N^Y_{n+1}} \delta_{Y^i_{n+1}}$.

In its original form, the Bernoulli filter jointly propagates the probability existence of the target and the distribution of the target state [24]. Combining the probability of existence and the state distribution into a single measure, it can be shown that the Bernoulli filter satisfies the integral equation (1.1), with the probability of existence of the target given by the mass $\gamma_n(1)$ and the distribution of the target state given by the normalised measure $\eta_n = \gamma_n/\gamma_n(1)$. The integral operator for the Bernoulli filter takes the following form

$$Q_{n+1,\gamma_n}(x_n, dx_{n+1}) := \frac{s_n(x_n)g_n(x_n)M_{n+1}(x_n, dx_{n+1}) + (\gamma_n(1)^{-1} - 1)\mu_{n+1}(dx_{n+1})}{(1 - \gamma_n(1)) + \gamma_n(g_n)} \quad (1.8)$$

where g_n is a likelihood function given by

$$g_n(x_n) \quad : \quad = (1 - d_n(x_n)) + d_n(x_n)\mathcal{Y}_n\left(l_n(x_n, \cdot)/h_n\right) \quad (1.9)$$

1.1.2 PHD filtering

A more challenging problem arises when the number of targets varies randomly in time, obscured by clutter, detection uncertainty and data association uncertainty. Suppose that at a given time n there are N^X_n targets $(X^i_n)_{1\leq i\leq N^X_n}$ each taking values in some measurable state space E_n. A target X^i_n, at time n, survives to the next time step with probability $s_n(X^i_n)$ and evolves to a new state according to a given elementary Markov transition M'_{n+1} from E_n into E_{n+1}. In addition X^i_n can spawn new targets at the next time, usually modelled by a spatial Poisson process with intensity measure $B_{n+1}(X^i_n, \cdot)$ on E_{n+1}. At the same time, an independent collection of new targets is added to the current configuration. This additional and spontaneous branching process is often modeled by a spatial Poisson process with a prescribed intensity measure μ_{n+1} on E_{n+1}. Each target X^i_{n+1} generates with probability $d_{n+1}(X^i_{n+1})$ an observation Y^i_{n+1} on some auxiliary state space, say E^Y_{n+1}, with probability density function $g_{n+1}(X^i_{n+1}, y)$. In addition to this partial observation point process we also observe an additional and independent Poisson point process with intensity function h_n. Multi-target tracking concerns the estimation of the random measures $\mathcal{X}_{n+1} = \sum_{1\leq i\leq N^X_n} \delta_{X^i_n}$, given the observation occupation measures $\mathcal{Y}_p = \sum_{1\leq i\leq N^Y_p} \delta_{Y^i_p}$.

The multi-target tracking problem is computationally intractable in general and the Probability Hypothesis Density PHD (filter), is an approximation that propagates the first-order statistical moment, or intensity, of the multi-target state forward in time [14]. The PHD filter satisfies the integral equation (1.1), with the integral operator given below

$$Q_{n+1,\gamma_n}(x_n, dx_{n+1}) = g_{n,\gamma_n}(x_n)M_{n+1}(x_n, dx_{n+1}) + \gamma_n(1)^{-1}\ \mu_{n+1}(dx_{n+1}) \quad (1.10)$$

where M_{n+1} is a Markov kernel defined by

$$M_{n+1}(x_n, dx_{n+1}) := \frac{s_n(x_n) M'_{n+1}(x_n, dx_{n+1}) + B_{n+1}(x_n, dx_{n+1})}{s_n(x_n) + b_n(x_n)} \tag{1.11}$$

with the branching rate $b_n(x_n) = B_{n+1}(1)(x_n)$. The likelihood function g_{n,γ_n} is given by

$$g_{n,\gamma_n} := r_n \times \widehat{g}_{n,\gamma_n} \quad \text{with} \quad r_n := (s_n + b_n) \tag{1.12}$$

and

$$\widehat{g}_{n,\gamma_n}(x_n) := (1 - d_n(x_n)) + d_n(x_n) \int \mathcal{Y}_n(dy) \, \frac{g_n(x_n, y_n)}{h_n(y_n) + \gamma_n(d_n g_n(., y_n))} \tag{1.13}$$

Since its inception by Mahler [14] in 2003, the PHD filter has attracted substantial interest to date. The development of numerical solutions for the PHD filter [21], [23] have opened the door to numerous novel extensions and applications. More details on the derivation of the PHD filter using random finite sets, Poisson techniques or random measures theoretic approaches can be found in the series of articles [4, 14, 19].

1.2 Statement of the main results

To describe with some conciseness the main result of this article, we need to introduce some notation. We let $\mathrm{Osc}_1(E_n)$, be the set of $\mathcal{E}_n$-measurable functions f on E_n with oscillations $\mathrm{osc}(f) = \sup_{x,x'} |f(x) - f(x')| \leq 1$. We denote by $\mu(f) = \int \mu(dx)\ f(x)$ the Lebesgue integral of f w.r.t. some measure $\mu \in \mathcal{M}(E_n)$, and we let $\|\mu - \nu\|_{\rm tv}$ be the total variation distance between two probability measures ν and μ on E_n.

We assume that the following pair of regularity conditions are satisfied.

(H_1) *: There exists a series of compact sets $I_n \subset (0, \infty)$ such that the initial mass value $\gamma_0(1) \in I_0$, and for any $m \in I_n$ $\eta \in \mathcal{P}(E_n)$, we have*

$$\theta_{-,n}(m) \leq \eta\left(G_{n,m\eta}\right) \leq \theta_{+,n}(m) \quad \textit{for some pair of positive functions } \theta_{+/-,n}.$$

The main implication of condition (H_1) comes from the fact that the total mass processes $\gamma_n(1)$ and their N-approximation models $\gamma_n^N(1)$ are finite and they evolves at every time n in a series of compact sets

$$I_n \subset [m_n^-, m_n^+] \subset (0, \infty)$$

with the sequence of parameters $m_n^{+/-}$ defined by the recursive equations $m_{n+1}^- = m_n^- \theta_{-,n}(m_n^-)$ and $m_{n+1}^+ = m_n^+ \theta_{+,n}(m_n^+)$, with the initial conditions $m_0^- = m_0^+ = \gamma_0(1)$.

(H_2) *: For any $n \geq 1$, $f \in \mathrm{Osc}_1(E_n)$, and any $(m, \eta), (m', \eta') \in (I_n \times \mathcal{P}(E_n))$, the one step mappings $\Gamma_n = \left(\Gamma_n^1, \Gamma_n^2\right)$ defined in (1.2) satisfy the following Lipschitz type inequalities:*

$$\left|\Gamma_n^1(m, \eta) - \Gamma_n^1(m', \eta')\right| \leq c(n)\ |m - m'| + \int \left|[\eta - \eta'](\varphi)\right|\ \Sigma^1_{n,(m',\eta')}(d\varphi) \tag{1.14}$$

$$\left|\left[\Gamma_n^2(m, \eta) - \Gamma_n^2(m', \eta')\right](f)\right| \leq c(n)\ |m - m'| + \int \left|[\eta - \eta'](\varphi)\right|\ \Sigma^2_{n,(m',\eta')}(f, d\varphi) \tag{1.15}$$

for some finite constants $c(n) < \infty$*, and some collection of bounded measures* $\Sigma^1_{n,(m',\eta')}$ *and* $\Sigma^2_{n,(m',\eta')}(f,\,.)$ *on* $\mathcal{B}(E_n)$ *such that*

$$\int \operatorname{osc}(\varphi)\ \Sigma^1_{n,(m,\eta)}(d\varphi) \leq \delta\left(\Sigma^1_n\right) \quad \textit{and} \quad \int \operatorname{osc}(\varphi)\ \Sigma^2_{n,(m,\eta)}(f,d\varphi) \leq \delta\left(\Sigma^2_n\right)$$

for some finite constant $\delta\left(\Sigma^i_n\right) < \infty$*,* $i = 1,2$*, whose values do dot depend on the parameters* $(m,\eta) \in (I_n \times \mathcal{P}(E_n)$ *and* $f \in \operatorname{Osc}_1(E_n)$*.*

Condition (H_2) is a rather basic and weak continuity type property. It states that the one step transformations of the flow of measures (1.2) are weakly Lipschitz, in the sense that the mass variations and the integral differences w.r.t. some test function f can be controlled by the different initial masses and measures w.r.t. a collection of integrals of a possibly infinite number of test functions. It is satisfied for a large class of one step transformations Γ_n. In section 2.3, we will verify that it is satisfied for the general class of Bernoulli and the PHD filters discussed in section 1.1.1 and section 1.1.2.

We are now in position to state the main results of this article. The first one is concerned with the exponential stability properties of the semigroup $\Gamma_{p,n} = \left(\Gamma^1_{p,n}, \Gamma^2_{p,n}\right)$, with $0 \leq p \leq n$ associated with the one step transformations of the flow (1.2). A more precise description and the complete proof of the next theorem is provided in section 3.

Theorem 1.1 *We let* $\Phi^1_{p,n,\nu}$ *and* $\Phi^2_{p,n,m}$ *be the semigroups associated with the one step transformations of the flow of total masses* $\Phi^1_{n,\nu_{n-1}} := \Gamma^1_n\left(\,.\,,\nu_{n-1}\right)$ *and measures* $\Phi^2_{n,m_{n-1}} := \Gamma^2_n\left(m_{n-1},\,.\right)$*, with a fixed collection of measures* $\nu := (\nu_n)_{n\geq 0} \in \prod_{n\geq 0}\mathcal{P}(E_n)$ *and masses* $m := (m_n)_{n\geq 0} \in \prod_{n\geq 0} I_n$*. When these semigroups are exponentially stable (in the sense that they forget exponentially fast their initial conditions) and when the pair of mappings* $\nu_{n-1} \mapsto \Phi^1_{n,\nu_{n-1}}$ *and* $m_{n-1} \mapsto \Phi^2_{n,m_{n-1}}$ *are sufficiently regular then we have the following contraction inequalities*

$$\left|\Gamma^1_{p,n}(u',\eta') - \Gamma^1_{p,n}(u,\eta)\right| \vee \left\|\Gamma^2_{p,n}(u',\eta') - \Gamma^2_{p,n}(u,\eta)\right\|_{\rm tv} \leq c\ e^{-\lambda(n-p)}$$

for any $p \leq n$*,* $u,u' \in I_p$*,* $\eta,\eta' \in \mathcal{P}(E_p)$*, and some finite constants* $c < \infty$ *and* $\lambda > 0$ *whose values do not depend on the time parameters* $p \leq n$*.*

The second theorem is concerned with estimating the approximation error associated with a N-approximation model satisfying condition (1.6). The first part of the theorem is proved in section 1.2. The proof of the uniform estimates is discussed in section 3.1 (see for instance lemma 3.4).

Theorem 1.2 *Under the assumptions* (H_1) *and* (H_2)*, the semigroup* $\Gamma_{p,n}$ *satisfies the same Lipschitz type inequalities as those stated in (1.14) and (1.15) for some collection of measures* $\Sigma^1_{p,n}$ *and* $\Sigma^2_{p,n}(f,\,.)$ *on* $\mathcal{B}(E_p)$*. In addition, for any* N*-approximation model satisfying condition (1.6) we have the estimates:*

$$\mathbb{E}\left(\left|V^{\gamma,N}_n(1)\right|^r\right)^{\frac{1}{r}} \leq a_r \sum_{p=0}^n \delta\left(\Sigma^1_{p,n}\right) \quad \textit{and} \quad \mathbb{E}\left(\left|V^{\eta,N}_n(f)\right|^r\right)^{\frac{1}{r}} \leq a_r \sum_{p=0}^n \delta\left(\Sigma^2_{p,n}\right) \tag{1.16}$$

for any $r \geq 1$*, and* $N \geq 1$*, with some constants* $a_r < \infty$ *whose values only depend on* r*. Furthermore, under the regularity conditions of theorem 1.1 the couple of estimates stated above are uniform w.r.t. the time horizon; that is, we have that* $\sup_{n\geq 0}\sum_{p=0}^n \delta\left(\Sigma^i_{p,n}\right) < \infty$*, for any* $i = 1,2$*.*

These rather abstract theorems apply to a general class of discrete generation measure-valued equations of the form (1.1). We illustrate the application of this pair of theorems in the analysis of the stability properties and the approximation convergence of the pair of multiple target filters presented in this introductory section. These results can basically be stated as follows:

- The Bernoulli filter presented in section 1.1.1 with a sufficiently mixing prediction and almost equal survival and spontaneous births rates $s_n \sim \mu_n(1)$ is exponentially stable.
- The PHD filter presented in section 1.1.2 is exponentially stable for small clutter intensities and sufficiently high detection probability and spontaneous birth rates.
- In both situations, the estimation error of any N-approximation model satisfying condition (1.6) does not accumulate over time. Furthermore, the uniform rates of convergence provided in theorem 1.2 allows to design stochastic algorithms with prescribed performance index at any time horizon.

We end this section with some direct consequences of theorem 1.2:

Firstly, we observe that the mean error estimates stated in the above theorem clearly implies the almost sure convergence results

$$\lim_{N\to\infty} \eta_n^N(f) = \eta_n(f) \quad \text{and} \quad \lim_{N\to\infty} \gamma_n^N(f) = \gamma_n(f)$$

for any bounded function f on E_n. Furthermore, with some information on the constants a_r, these $\mathbb{L}_r$-mean error bounds can be turned to exponential concentration inequalities. To be more precise, by lemma 7.3.3 in [5], the collection of constants a_r in theorem 1.2, can be chosen so that

$$a_{2r}^{2r} \leq b^{2r}\ (2r)!\ 2^{-r}/r! \quad \text{and} \quad a_{2r+1}^{2r+1} \leq b^{2r+1}(2r+1)!\ 2^{-r}/r! \tag{1.17}$$

for some $b < \infty$, whose values do not depend on r. Using the above $\mathbb{L}_r$-mean error bounds we can establish the following non asymptotic Gaussian tail estimates:

$$\mathbb{P}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right| \geq \frac{b_n}{\sqrt{N}} + \epsilon\right) \leq \exp\left(-\frac{N\epsilon^2}{2b_n^2}\right) \quad \text{with} \quad b_n \leq b\ \sum_{p=0}^{n} \delta\left(\Sigma_{p,n}^2\right)$$

The above result is a direct consequence of the following observation

$$\forall r \geq 1 \qquad \mathbb{E}\left(U^r\right)^{\frac{1}{r}} \leq a_r\ b \Rightarrow \mathbb{P}\left(U \geq b+\epsilon\right) \leq \exp\left(-\epsilon^2/(2b)\right)$$

for any non negative random variable U. To check this claim, we use the following Laplace estimate

$$\forall t \geq 0 \quad \mathbb{E}\left(e^{tU}\right) \quad \leq \quad \exp\left(\frac{(bt)^2}{2} + bt\right) \Rightarrow \mathbb{P}\left(U \geq b+\epsilon\right) \leq \exp\left(-\sup_{t\geq 0}\left(\epsilon t - \frac{(bt)^2}{2}\right)\right)$$

It is worth noting that the above constructions allows us to consider with further work branching particle models in path spaces. These path space models arise in the analysis of the historical process associated with a branching models as well as the analysis of a filtering problem of the whole signal path given a series of observations. For instance, let us suppose that the Markov transitions M_n defined in (1.10 are the elementary transition of a Markov chain of the following form

$$X_n := \left(X'_p\right)_{0\leq p\leq n} \in E_n := \prod_{0\leq p\leq n} E'_p$$

In other words X_n represents the paths from the origin up to the current time of an auxiliary Markov chain X'_n taking values in some measurable state spaces E'_n, with Markov transitions M'_n. We assume that the potential functions g_{n,γ_n} only depend on the terminal state of the path, in the sense that $g_{n,\gamma_n}(X_n) = g'_{n,\gamma_n}(X'_n)$, for some potential function g'_{n,γ_n} on E'_n. In multiple target tracking problems, these path space models provide a way to estimate the conditional intensity of the path of a given target in a multi-target environment related to some likelihood function that only depends on the terminal state of the signal path.

In practice, it is essential to observe that the mean field particle interpretations of these path space models simply consist of keeping track of the whole history of each particle. It can be shown that the resulting particle model can be interpreted as the genealogical tree model associated with a genetic type model (see for instance [5]). In this situation, η_n^N is the occupation measure of a random genealogical tree, each particle represents the ancestral lines of the current individuals.

We end this section with some standard notation used in the paper:

We denote respectively by $\mathcal{M}(E)$, $\mathcal{P}(E)$, and $\mathcal{B}(E)$, the set of all finite positive measures μ on some measurable space $(E,\mathcal{E})$, the convex subset of all probability measures, and the Banach space of all bounded and measurable functions f equipped with the uniform norm $\|f\|$. We denote by f^- and f^+ the infimum and the supremum of a function f. For measurable subsets $A\in\mathcal{E}$, in various instances we slightly abuse notation and we denote $\mu(A)$ instead of $\mu(1_A)$; and we set δ_a the Dirac measure at $a\in E$. We recall that a bounded and positive integral operator Q from a measurable space $(E_1,\mathcal{E}_1)$ into an auxiliary measurable space $(E_2,\mathcal{E}_2)$ is an operator $f\mapsto Q(f)$ from $\mathcal{B}(E_2)$ into $\mathcal{B}(E_1)$ such that the functions

$$x\mapsto Q(f)(x) := \int_{E_2} Q(x,dy)f(y)$$

are $\mathcal{E}_1$-measurable and bounded for some measures $Q(x,.)\in\mathcal{M}(E_2)$. These operators also generate a dual operator $\mu\mapsto\mu Q$ from $\mathcal{M}(E_1)$ into $\mathcal{M}(E_2)$ defined by $(\mu Q)(f) := \mu(Q(f))$. A Markov kernel is a positive and bounded integral operator M with $M(1)=1$. We denote by $Q_{p,n} = Q_{p+1}Q_{p+2}\ldots Q_n$, with $p\leq n$ the semigroup associated with a given sequence of bounded and positive integral operator Q_n from some measurable spaces $(E_{n-1},\mathcal{E}_{n-1})$ into $(E_n,\mathcal{E}_n)$. For $p=n$, we use the convention $Q_{n,n}=Id$, the identity operator.

We associate with a bounded positive potential function $G : x\in E\mapsto G(x)\in[0,\infty)$, the Bayes-Boltzmann-Gibbs transformations

$$\Psi_G \ : \ \eta\in\mathcal{M}(E)\mapsto\Psi_G(\eta)\in\mathcal{P}(E) \quad \text{with} \quad \Psi_G(\eta)(dx) := \frac{1}{\eta(G)}\, G(x)\,\eta(dx)$$

provided $\eta(G)>0$. We recall that $\Psi_G(\eta)$ can be expressed in terms of a Markov transport equation

$$\eta S_\eta = \Psi_G(\eta) \tag{1.18}$$

for some selection type transition $S_\eta(x,dy)$. For instance, we can take

$$S_\eta(x,dy) := \frac{\epsilon}{\eta(G)}\ \delta_x(dy) + \left(1-\frac{\epsilon}{\eta(G)}\right)\ \Psi_{(G-\epsilon)}(\eta)(dy) \tag{1.19}$$

for any $\epsilon \geq 0$ s.t. $G(x)\geq\epsilon$. Notice that for $\epsilon=0$, we have $S_\eta(x,dy)=\Psi_G(\eta)(dy)$. We can also choose

$$S_\eta(x,dy) := \epsilon G(x)\ \delta_x(dy) + (1-\epsilon G(x))\ \ \Psi_G(\eta)(dy) \tag{1.20}$$

for any $\epsilon\geq 0$ that may depend on the current measure η, and s.t. $\epsilon G(x)\leq 1$. For instance, we can choose $1/\epsilon$ to be the η-essential maximum of the potential function G. Finally, in the context of Bernoulli and PHD filtering we set $\overline{\mu}_{n+1}=\mu_{n+1}/\mu_{n+1}(1)$, for any $n\geq 0$, the normalized spontaneous birth measures.

2 Semigroup description

2.1 The Bernoulli filter semigroup

By construction, we notice that the mass process and the normalized measures are given by the rather simple recursive formulae

$$\gamma_{n+1}(1) = \frac{\gamma_n(1)\eta_n(g_n)}{(1-\gamma_n(1))+\gamma_n(1)\eta_n(g_n)}\ \Psi_{g_n}(\eta_n)(s_n) + \frac{(1-\gamma_n(1))}{(1-\gamma_n(1))+\gamma_n(1)\eta_n(g_n)}\ \mu_{n+1}(1) \tag{2.1}$$

and

$$\eta_{n+1} := \alpha_n(\gamma_n)\ \Psi_{g_n s_n}(\eta_n)M_{n+1} + (1-\alpha_n(\gamma_n))\ \overline{\mu}_{n+1}$$

with the mappings $\alpha_n\ :\ \gamma\in\mathcal{M}(E_n)\mapsto\alpha_n(\gamma)\in[0,1]$ defined by

$$\alpha_n(\gamma) = \frac{\gamma(g_ns_n)}{\gamma(s_ng_n)+(1-\gamma(1))\mu_{n+1}(1)}$$

By construction, if we set $\gamma = m\times\eta$ then

$$\begin{aligned}
\Gamma^1_{n+1}(m,\eta) &= \frac{\gamma(g_n)}{(1-m)+\gamma(g_n)}\ \Psi_{g_n}(\eta)(s_n) + \frac{(1-m)}{(1-m)+\gamma(g_n)}\ \mu_{n+1}(1)\\
\Gamma^2_{n+1}(m,\eta) &= \Psi_{g_ns_n}(\eta)M_{n+1,\gamma}
\end{aligned}$$

with the collection of Markov transitions $M_{n+1,\gamma}$ defined below

$$M_{n+1,\gamma}(x,.) := \alpha_n(\gamma)\,M_{n+1}(x,.) + (1-\alpha_n(\gamma))\ \ \overline{\mu}_{n+1} \tag{2.2}$$

Next we provide an alternative interpretation of the mapping Γ^2_{n+1}. Firstly, observe that

$$\Psi_{g_ns_n}(\eta)M_{n+1,\gamma}(f) = \frac{\eta\left(Q_{n+1,m}(f)\right)}{\eta\left(Q_{n+1,m}(1)\right)} \tag{2.3}$$

with the integral operator

$$Q_{n+1,m}(f)(x) := mg_n(x)s_n(x)M_{n+1}(f)(x) + (1-m)\ \mu_{n+1}(f)$$

This implies that

$$\Gamma^2_{n+1}(m,\eta) = \Psi_{\widehat{G}_{n,m}}(\eta)\widehat{M}_{n+1,m}$$

with the potential function

$$\widehat{G}_{n,m} = mg_ns_n + (1-m)\ \mu_{n+1}(1) \tag{2.4}$$

and the Markov transitions

$$\widehat{M}_{n+1,m}(f) := \frac{mg_ns_n}{mg_ns_n + (1-m)\ \mu_{n+1}(1)}\ M_{n+1}(f) + \frac{(1-m)\mu_{n+1}(1)}{mg_ns_n + (1-m)\ \mu_{n+1}(1)}\ \overline{\mu}_{n+1}(f) \tag{2.5}$$

The condition (H_1) is clearly not met for the Bernoulli filter (1.8) when $s_n = 0$ and $\mu_{n+1}(1) = 0$, since in this situation $\gamma_n = 0$ for any $n \geq 1$. Nevertheless, this condition is met with $I_n \subset (0,1]$ and $m\theta_{+,n}(m) = 1$, as long as s_n and $\mu_{n+1}(1)$ are uniformly bounded from below. It is also met for $s_n = 0$, as long as $0 < \mu_{n+1}(1) < 1$ and the likelihood function given in (1.9) is uniformly bounded. The condition is also met for $\mu_{n+1}(1) = 0$, as long as $\gamma_0(1) > 0$, and the likelihood function given in (1.9) and the function s_n are uniformly lower bounded.

We prove these assertions using the fact that

$$\gamma_{n+1}(1) = \widehat{\gamma}_n(1)\ \Psi_{g_n}(\eta_n)(s_n) + (1-\widehat{\gamma}_n(1))\ \mu_{n+1}(1) \tag{2.6}$$

with the updated mass parameters $\widehat{\gamma}_n(1) \in [0,1]$ given below

$$\widehat{\gamma}_n(1) := \frac{\gamma_n(1)\eta_n(g_n)}{(1-\gamma_n(1)) + \gamma_n(1)\eta_n(g_n)}$$

If we set $s_n^- := \inf_{E_n} s_n$ and $s_n^+ = \sup_{E_n} s_n$ then

$$\forall n \geq 1 \qquad \gamma_n(1) \in \left[m_n^-, m_n^+\right]$$

with parameters

$$m_n^- = \mu_n(1) \wedge s_{n-1}^- \quad \text{and} \quad m_n^+ = \mu_n(1) \vee s_{n-1}^+\ \ (\leq 1)$$

If s_n and $\mu_{n+1}(1)$ are uniformly bounded from below then we have $m_n^- > 0$. In addition, for the constant mapping $s_n = \mu_{n+1}(1)$, the total mass process is constant

$$\gamma_{n+1}(1) = m_{n+1}^+ = m_{n+1}^- = \mu_{n+1}(1)$$

for any $n \geq 0$. Furthermore, in this situation the flow of normalized measures is given by the updating-prediction transformation defined by

$$\forall n \geq 0 \qquad \eta_{n+1} = \Psi_{g_n^{(s)}}(\eta_n)\, M_{n+1}^{(s)}$$

with the likelihood function $g_n^{(s)}$ and the Markov transitions $M_{n+1}^{(s)}$ defined by

$$g_n^{(s)} := s_ng_n + (1-s_n) \quad \text{and} \quad M_{n+1}^{(s)}(f) := \frac{s_ng_nM_{n+1}(f) + (1-s_n)\ \overline{\mu}_{n+1}(f)}{s_ng_n + (1-s_n)} \tag{2.7}$$

When $\mu_{n+1}(1) = 0$, the flow of normalized measures is again given by a simple updating-prediction equation

$$\eta_{n+1} = \Psi_{g_n s_n}(\eta_n) M_{n+1} \quad \text{and} \quad \gamma_{n+1}(1) = \Psi_{g_n}(\eta_n)(s_n) \times \theta_{\eta_n(g_n)}(\gamma_n(1)) \tag{2.8}$$

with the increasing mappings θ_a defined below

$$x \in [0,1] \mapsto \theta_a(x) := ax/[ax + (1-x)] \tag{2.9}$$

In addition, if $s_n^- > 0$ then

$$m_{n+1}^- \geq s_n^- \times \frac{g_n^- m_n^-}{g_n^- m_n^- + (1 - m_n^-)} > 0$$

as long as $g_n^- := \inf_{E_n} g_n > 0$, and $\gamma_0(1) > 0$. We prove this inequality using the fact that the mapping $(a, x) \in [0, \infty[\times[0, 1] \mapsto \theta_a(x)$ is increasing in both coordinates. In the case where $s_n = 1$, using the fact that and $\theta_a \circ \theta_b = \theta_{ab}$, we prove that

$$\gamma_{n+1}(1) = \theta_{\eta_n(g_n)}(\gamma_n(1)) = \theta_{\prod_{p=0}^n \eta_p(g_p)}(\gamma_0(1))$$

Conversely, when $\gamma_0(1) < 1$ and $0 < \mu_{n+1}(1) < 1$ and $s_n = 0$, for any $n \geq 0$, then we have a constant flow of normalized measures

$$\forall n \geq 1 \qquad \eta_n = \overline{\mu}_n$$

and the total mass process is such that

$$\gamma_n(1) \in]0,1[\implies \gamma_{n+1}(1) = \mu_{n+1}(1) \times \left[1 - \theta_{\overline{\mu}_n(g_n)}(\gamma_n(1))\right] \in]0,1[$$

with the convention $\overline{\mu}_0 = \eta_0$, for $n = 0$. In addition, if $\mu_{n+1}(1) = 1$ then we have

$$\gamma_{2(n+1)}(1) = \theta_{\prod_{p=0}^n (b_{2p}/b_{2p+1})}(\gamma_0(1)) \quad \text{and} \quad \gamma_{2n+1}(1) = \theta_{b_{2n}^{-1} \prod_{p=0}^{n-1} (b_{2p+1}/b_{2p})}(\gamma_0(1))$$

for any $n \geq 0$, with the parameters $b_n := \overline{\mu}_n(g_n)$. We prove these formuae using the the fact that $1 - \theta_a(x) = \theta_{1/a}(1-x)$, and $\theta_a \circ \theta_b = \theta_{ab}$. This again implies that $m_n^- > 0$ as long as $\gamma_0(1) > 0$ and the likelihood function are uniformly lower bounded.

2.2 The PHD filter semigroup

By construction, if we set $\gamma = m \times \eta$ then we find that

$$\Gamma_{n+1}^1(m, \eta) = \gamma(g_{n,\gamma}) + \mu_{n+1}(1) \quad \text{and} \quad \Gamma_{n+1}^2(m, \eta) = \Psi_{g_{n,\gamma}}(\eta) M_{n+1,\gamma}$$

In the above display, $M_{n+1,\gamma}$ is the collection of Markov transitions defined below

$$M_{n+1,\gamma}(x, .) := \alpha_n(\gamma) M_{n+1}(x, .) + (1 - \alpha_n(\gamma)) \; \overline{\mu}_{n+1} \quad \text{with} \quad \alpha_n(\gamma) = \frac{\gamma(g_{n,\gamma})}{\gamma(g_{n,\gamma}) + \mu_{n+1}(1)}$$

The interpretation of the updating transformation $\Psi_{g_{n,\gamma}}(\eta)$ in terms of a Markov transport equation is non unique. For instance, using (1.12) this Bolzmann-Gibbs transformation can be decomposed into two parts. The first one relates to the undetectable targets and the second is associated with non clutter observations. An alternative description is provided below. We consider a virtual auxiliary observation point c (corresponding to undetectable targets) and set $\mathcal{Y}_n^c = \mathcal{Y}_n + \delta_c$. We also denote by $g_{n,\gamma}^c(.,y)$ the function defined below

$$g_n^\gamma(.,y) = \begin{cases} r_n(1-d_n) & \text{if} \quad y = c \\ r_n \ \dfrac{d_n g_n(.,y_n)}{h_n(y) + \gamma(d_n g_n(.,y))} & \text{if} \quad y \neq c \end{cases}$$

In this notation, the updating transformation $\Psi_{g_{n,\gamma}}(\eta)$ can be rewritten in the following form

$$\Psi_{g_{n,\gamma}}(\eta) = \Psi_{\overline{g}_{n,\gamma}}(\eta) \quad \text{with} \quad \overline{g}_{n,\gamma} = \int \mathcal{Y}_n^c(dy)\ g_n^\gamma(.,y)$$

The averaged potential function $\overline{g}_{n,\gamma}$ allows us to measure the likelihood of signal states w.r.t. the current observation measure $\mathcal{Y}_n^c$. Using (1.18), the Bolzmann-Gibbs transformation $\Psi_{\overline{g}_{n,\gamma}}(\eta)$ can be interpreted as non linear Markov transport equation of the following form

$$\Psi_{\overline{g}_{n,\gamma}}(\eta) = \eta S_{n,\gamma} \quad \text{and} \quad \Gamma^2(m,\eta) = \eta K_{n+1,\gamma} \quad \text{with} \quad K_{n+1,\gamma} = S_{n,\gamma} M_{n+1,\gamma} \tag{2.10}$$

for some Markov transitions $S_{n,\gamma}$ from E_n into itself.

We also notice that condition (H_1) holds as long as the functions s_n, b_n, and $g_n(.,y_n)$ are uniformly bounded and $\mu_n(1) > 0$. It is also met when $\mu_n(1) = 0$, as long as $r_n = (s_n + b_n)$ is uniformly lower bounded and $\mathcal{Y}_n \neq 0$ or $d_n < 1$.

2.3 Lipschitz regularity properties

Firstly, we mention that condition (H_2) can be replaced by the following regularity condition:

(H_2') : *For any $n \geq 1$, $f \in \mathrm{Osc}_1(E_n)$, and any $(m,\eta),(m',\eta') \in (I_n \times \mathcal{P}(E_n))$, the integral operators $Q_{n,m\eta}$ satisfy the following Lipschitz type inequalities:*

$$\left\|Q_{n,m\eta}(f) - Q_{n,m'\eta'}(f)\right\| \leq c(n)\ |m-m'| + \int \ \left|[\eta-\eta'](\varphi)\right|\ \Sigma_{n,(m',\eta')}(f,d\varphi) \tag{2.11}$$

for some collection of bounded measures $\Sigma_{n,(m',\eta')}(f,.)$ on $\mathcal{B}(E_n)$ such that

$$\int \ \mathrm{osc}(\varphi)\ \Sigma_{n,(m,\eta)}(f,d\varphi) \leq \delta\left(\Sigma_n\right)$$

for some finite constant $\delta(\Sigma_n) < \infty$, whose values do dot depend on the parameters $(m,\eta) \in (I_n \times \mathcal{P}(E_n))$ and $f \in \mathrm{Osc}_1(E_n)$.

We prove $(H_2') \Rightarrow$ (1.14) using the decompositions

$$m\eta Q_{n,m\eta} - m'\eta' Q_{n,m'\eta'} = m\eta\left[Q_{n,m\eta} - Q_{n,m'\eta'}\right] + \left[m\eta - m'\eta'\right] Q_{n,m'\eta'}$$

and of course $[m\eta - m'\eta'] = [m-m']\,\eta + m'\,[\eta-\eta']$. To prove $(H_2') \Rightarrow$ (1.15), we let $\gamma = m\eta$ and $\gamma' = m'\eta'$ and we use the decomposition

$$\left[\Gamma_n^2(m,\eta) - \Gamma_n^2(m',\eta')\right](f) = \frac{1}{\gamma Q_{n,\gamma}(1)}\ \left[\gamma Q_{n,\gamma} - \gamma' Q_{n,\gamma'}\right]\left(f - \Gamma_n^2(m',\eta')(f)\right)$$

The Bernoulli filter (1.8) satisfies (H_2'), as long as the likelihood functions g_n given in (1.9) are uniformly bounded above. In this situation, (2.11) is met with

$$\left\|Q_{n,m\eta}(f) - Q_{n,m'\eta'}(f)\right\| \leq c(n)\ |m - m'| + c'(n)\left|[\eta - \eta'](g_n)\right|$$

for some finite constant $c'(n) < \infty$.

The PHD equation satisfies (H_2'), as long as the functions $h_n(y) + g'_{n,y}$ with $g'_{n,y} := d_n g_n(.,y)$ are uniformly bounded above and below. To prove this claim, we simply use the fact that

$$\left\|\widehat{g}_{n,\gamma} - \widehat{g}_{n,\gamma'}\right\| \leq c_n \left[|m' - m| + \int \mathcal{Y}_n(dy)\ \left|[\eta' - \eta](g'_{n,y})\right|\right]$$

This estimate is a direct consequence of the following one

$$\widehat{g}_{n,\gamma}(x) - \widehat{g}_{n,\gamma'}(x) = \int \mathcal{Y}_n(dy)\ \frac{g'_{n,y}(x)}{h_n(y) + \gamma(g'_{n,y})} \frac{[\gamma' - \gamma](g'_{n,y})}{h_n(y) + \gamma'(g'_{n,y})}$$

Next, we provide a pivotal regularity property of the semigroup $(\Gamma_{p,n})_{0\leq p\leq n}$ associated with the one step transformations of the flow (1.2).

Proposition 2.1 *We assume that conditions (H_1) and (H_2) are satisfied. Then, for any $0 \leq p \leq n$, $f \in \mathrm{Osc}_1(E_n)$, and any $(m,\eta),(m',\eta') \in (I_p \times \mathcal{P}(E_p))$, we have the following Lipschitz type inequalities:*

$$\begin{aligned}
\left|\Gamma^1_{p,n}(m,\eta) - \Gamma^1_{p,n}(m',\eta')\right| &\leq c_p(n)\ |m - m'| + \int \left|[\eta - \eta'](\varphi)\right|\ \Sigma^1_{p,n,(m',\eta')}(d\varphi)\\
\left|\left[\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m',\eta')\right](f)\right| &\leq c_p(n)\ |m - m'| + \int \left|[\eta - \eta'](\varphi)\right|\ \Sigma^2_{p,n,(m',\eta')}(f,d\varphi)
\end{aligned}$$

for some finite constants $c_p(n) < \infty$, and some collection of bounded measures $\Sigma^1_{p,n,(m',\eta')}$ and $\Sigma^2_{p,n,(m',\eta')}(f,\,.)$ on $\mathcal{B}(E_p)$ such that

$$\int \mathrm{osc}(\varphi)\ \Sigma^1_{p,n,(m,\eta)}(d\varphi) \leq \delta\left(\Sigma^1_{p,n}\right) \quad \text{and} \quad \int \mathrm{osc}(\varphi)\ \Sigma^2_{p,n,(m,\eta)}(f,d\varphi) \leq \delta\left(\Sigma^2_{p,n}\right) \qquad (2.12)$$

for some finite constant $\delta\left(\Sigma^i_{p,n}\right) < \infty$, $i = 1,2$, whose values do dot depend on the parameters $(m,\eta) \in (I_p \times \mathcal{P}(E_p)$ and $f \in \mathrm{Osc}_1(E_n)$.

Proof:
To prove this proposition, we use a backward induction on the parameter $1 \leq p \leq n$. For $p = (n-1)$, we have $\Gamma^i_{n-1,n} = \Gamma^i_n$, with $i = 1,2$, so that the desired result is satisfied for $p = (n-1)$. We further assume that the estimates hold at a given rank $p < n$. To prove the estimates at rank $(p-1)$, we recall that

$$\Gamma_{p-1,n}(m,\eta) = \Gamma_{p,n}\left(\Gamma_p(m,\eta)\right) \Rightarrow \forall i = 1,2 \qquad \Gamma^i_{p-1,n}(m,\eta) = \Gamma^i_{p,n}\left(\Gamma_p(m,\eta)\right)$$

Under the induction hypothesis

$$\begin{aligned}
\left|\Gamma^1_{p-1,n}(m,\eta) - \Gamma^1_{p-1,n}(m',\eta')\right| &= \left|\Gamma^1_{p,n}\left(\Gamma_p(m,\eta)\right) - \Gamma^1_{p,n}\left(\Gamma_p(m',\eta')\right)\right|\\
&\leq c_p(n)\ |\Gamma^1_p(m,\eta) - \Gamma^1_p(m',\eta')|\\
&\quad + \int \left|\left[\Gamma^2_p(m,\eta) - \Gamma^2_p(m',\eta')\right](\varphi)\right|\ \Sigma^1_{p,n,\Gamma_p(m',\eta')}(d\varphi)
\end{aligned}$$

On the other hand

$$|\Gamma^1_p(m,\eta)-\Gamma^1_p(m',\eta')|\leq c(p)\ |m-m'|+\int\ \left|[\eta-\eta'](\varphi)\right|\ \Sigma^1_{p,(m',\eta')}(d\varphi)$$

and

$$\left|\left[\Gamma^2_p(m,\eta)-\Gamma^2_p(m',\eta')\right](\varphi)\right|\leq c(p)\ |m-m'|+\int\ \left|[\eta-\eta'](\psi)\right|\ \Sigma^2_{p,(m',\eta')}(\varphi,d\psi)$$

The end of the proof is now clear. The analysis of $\Gamma^2_{p-1,n}$ follows the same line of arguments and is omitted. This ends the proof of the proposition. ∎

2.4 Proof of theorem 1.2

This section is mainly concerned with the proof of the couple of estimates (1.16) stated in theorem 1.2.

We use the decomposition

$$\begin{aligned}\left(\gamma^N_n(1),\eta^N_n\right)-(\gamma_n(1),\eta_n) &= \left[\Gamma_{0,n}\left(\gamma^N_0(1),\eta^N_0\right)-\Gamma_{0,n}\left(\gamma_0(1),\eta_0\right)\right]\\ &\quad+\sum_{p=1}^n\left[\Gamma_{p,n}\left(\gamma^N_p(1),\eta^N_p\right)-\Gamma_{p-1,n}\left(\gamma^N_{p-1}(1),\eta^N_{p-1}\right)\right]\end{aligned}\tag{2.13}$$

and the fact that

$$\Gamma_{p-1,p}\left(\gamma^N_{p-1}(1),\eta^N_{p-1}\right)=\left(\gamma^N_p(1),\Gamma^2_{p-1,p}\left(\gamma^N_{p-1}(1),\eta^N_{p-1}\right)\right)$$

to show that

$$\begin{aligned}\gamma^N_n(1)-\gamma_n(1) &= \left[\Gamma^1_{0,n}\left(\gamma^N_0(1),\eta^N_0\right)-\Gamma^1_{0,n}\left(\gamma_0(1),\eta_0\right)\right]\\ &\quad+\sum_{p=1}^n\left[\Gamma^1_{p,n}\left(\gamma^N_p(1),\eta^N_p\right)-\Gamma^1_{p,n}\left(\gamma^N_p(1),\Gamma^2_{p-1,p}\left(\gamma^N_{p-1}(1),\eta^N_{p-1}\right)\right)\right]\end{aligned}$$

Recalling that $\gamma^N_0(1)=\gamma_0(1)$, using proposition 2.1, we find that

$$\sqrt{N}\left|\gamma^N_n(1)-\gamma_n(1)\right|\leq\sum_{p=0}^n c_p(n)\ \int\ \left|[W^N_p(\varphi)\right|\ \Sigma^{(N,1)}_{p,n}(d\varphi)$$

with the predictable measure $\Sigma^{(N,1)}_{p,n}=\Sigma^1_{p,n,(m,\eta)}$ associated with the parameters $(m,\eta)=(\gamma^N_p(1),\Gamma^2_{p-1,p}\left(\gamma^N_{p-1}(1),\eta^N_{p-1}\right))$, with $0<p\leq n$; and for $p=0$, we set $\Sigma^{(N,1)}_{0,n}=\Sigma_{0,n,(\gamma_0(1),\eta_0)}$. Combing the generalized Minkowski's inequality with (1.6) we have

$$\mathbb{E}\left(\left|\int\ \left|W^N_p(\varphi)\right|\ \Sigma^{(N,1)}_{p,n}(d\varphi)\right|^r\left|\mathcal{F}^{(N)}_{p-1}\right.\right)^{\frac{1}{r}}\leq a_r\ \delta\left(\Sigma^1_{p,n}\right)$$

for some constants a_r whose values only depend on the time parameter. This clearly implies that

$$\mathbb{E}\left(\left|\gamma^N_n(1)-\gamma_n(1)\right|^r\right)^{\frac{1}{r}}\leq a_r\ \sum_{p=0}^n\delta\left(\Sigma^1_{p,n}\right)$$

The normalized occupation measures can be analyzed in the same way using the decomposition given below:

$$\begin{aligned}\eta_n^N - \eta_n &= \left[\Gamma^2_{0,n}\left(\gamma_0^N(1),\eta_0^N\right) - \Gamma^2_{0,n}\left(\gamma_0(1),\eta_0\right)\right] \\ &\qquad + \sum_{p=1}^{n}\left[\Gamma^2_{p,n}\left(\gamma_p^N(1),\eta_p^N\right) - \Gamma^2_{p,n}\left(\gamma_p^N(1),\eta_{p-1}^N K_{p,(\gamma_{p-1}^N(1),\eta_{p-1}^N)}\right)\right]\end{aligned}$$

This ends the proof of the theorem 1.2. ∎

3 Functional contraction inequalities

3.1 Stability properties

This section is concerned with the long time behavior of nonlinear measure-valued processes of the form (1.2). The complexity of these models depend in part on the interaction function between the flow of masses $\gamma_n(1)$ and the flow of probability measures $\eta_n = \gamma_n/\gamma_n(1)$. One natural way to start the analysis of these models is to study the stability properties of the measure-valued semigroup associated with a fixed flow of masses, and vice versa. These two mathematical objects are defined below.

Definition 3.1 *We associate with a flow of masses $m = (m_n)_{n\geq 0} \in \prod_{n\geq 0} I_n$ and probability measures $\nu := (\nu_n)_{n\geq 0} \in \prod_{n\geq 0} \mathcal{P}(E_n)$ the pair of semigroups*

$$\Phi^1_{p,n,\nu} := \Phi^1_{n,\nu_{n-1}} \circ \ldots \circ \Phi^1_{1,\nu_0} \quad \text{and} \quad \Phi^2_{p,n,m} := \Phi^2_{n,m_{n-1}} \circ \ldots \circ \Phi^2_{1,m_0} \tag{3.1}$$

with $0 \leq p \leq n$, and the one step transformations

$$\begin{aligned}\Phi^1_{n,\nu_{n-1}} &: u \in I_{n-1} \mapsto \Phi^1_{n,\nu_{n-1}}(u) := \Gamma^1_n\left(u,\nu_{n-1}\right) \in I_n \\ \Phi^2_{n,m_{n-1}} &: \eta \in \mathcal{P}(E_{n-1}) \mapsto \Phi^2_{n,m_{n-1}}(\eta) := \Gamma^2_n\left(m_{n-1},\eta\right) \in \mathcal{P}(E_n)\end{aligned}$$

By construction, using a simple induction on the time parameter n, we find that

$$(m_0,\nu_0) = (\gamma_0(1),\eta_0) \quad \text{and} \quad \forall n\geq 1 \quad m_n = \Phi^1_{n,\nu_{n-1}}(m_{n-1}) \quad \text{and} \quad \nu_n = \Phi^2_{n,m_{n-1}}(\nu_{n-1})$$

$$\Updownarrow$$

$$\forall n \geq 0 \quad (m_n,\nu_n) = (\gamma_n(1),\eta_n)$$

In the cases that are of particular interest, the semigroups $\Phi^1_{p,n,\nu}$ and $\Phi^2_{p,n,m}$ will have a Feynman-Kac representation. These models are rather well understood. A brief review on their contraction properties is provided in section 3.2. Further details can be found in the monograph [5]. The first basic regularity property of these models which are needed is the following weak Lipschitz type property :

(Lip(Φ)) For any $p \leq n$, $u,u' \in I_p$, $\eta,\eta' \in \mathcal{P}(E_p)$ and $f \in Osc_1(E_n)$ the following Lipschitz inequalities

$$\left|\Phi^1_{p,n,\nu}(u) - \Phi^1_{p,n,\nu}(u')\right| \leq a^1_{p,n}\ |u-u'| \tag{3.2}$$

$$\left|\left[\Phi^2_{p,n,m}(\eta) - \Phi^2_{p,n,m}(\eta')\right](f)\right| \leq a^2_{p,n}\int \left|[\eta-\eta'](\varphi)\right|\ \Omega^2_{p,n,\eta'}(f,d\varphi) \tag{3.3}$$

for some finite constants $a^i_{p,n} < \infty$*, with* $i = 1, 2$*, and some collection of Markov transitions* $\Omega^2_{p,n,\eta'}$ *from* $Osc_1(E_n)$ *into* $Osc_1(E_p)$*, with* $p \leq n$*, whose values only depend on the parameters* p, n*, resp.* p, n *and* η'*.*

The semigroups $\Phi^1_{p,n,\nu}$ and $\Phi^2_{p,n,m}$ may or may bot be asymptotically stable depending on whether $a^i_{p,n}$ tends to 0, as $(n-p) \to \infty$. In section 3.3 we provide a set of easily checked regularity conditions under which the semigroups associated with the Bernoulli models discussed in 2.1 are asymptotically stable.

The second step in the study of the stability properties of the semigroups associated with the flow (1.2) is the following continuity property:

$(Cont(\Phi))$ *For any* $n \geq 1$*,* $u, u' \in I_{n-1}$*,* $\eta, \eta' \in \mathcal{P}(E_{n-1})$ *and any* $f \in Osc_1(E_n)$

$$\left|\Phi^1_{n,\eta}(u) - \Phi^1_{n,\eta'}(u)\right| \leq \tau^1_n \int \left|[\eta - \eta'](\varphi)\right| \, \Omega^1_{n,\eta'}(d\varphi) \tag{3.4}$$

$$\left|\left[\Phi^2_{n,u}(\eta) - \Phi^2_{n,u'}(\eta)\right](f)\right| \leq \tau^2_n \, |u - u'| \tag{3.5}$$

for some finite constants $\tau^i_n < \infty$*, with* $i = 1, 2$*, and some collection probability measures* $\Omega^1_{n,\nu'}$ *on* $Osc_1(E_{n-1})$*, whose values only depend on the parameters* n*, resp.* n *and* ν'*.*

This elementary continuity condition allows us to enter the contraction properties of the semigroups $\Phi^1_{p,n,\nu}$ and $\Phi^2_{p,n,m}$ in the stability analysis of the flow of measures (1.2). The resulting functional contraction inequalities will be described in terms of the following collection of parameters.

Definition 3.2 *When the couple of conditions* $(Lip(\Phi))$ *and* $(Cont(\Phi))$ *stated above are satisfied, for any* $i = 1, 2$ *and* $p \leq n$ *we set*

$$\overline{a}^i_{p,n} = \tau^i_{p+1} \; a^i_{p+1,n} \qquad b_{p,n} = \sum_{p<q<n} \overline{a}^1_{p,q} \; \overline{a}^2_{q,n} \quad \text{and} \quad b'_{p,n} = \sum_{p \leq q < n} a^1_{p,q} \; \overline{a}^2_{q,n} \tag{3.6}$$

The main result of this section is the following proposition.

Proposition 3.3 *If conditions* $(Lip(\Phi))$ *and* $(Cont(\Phi))$ *are satisfied, then for any* $p \leq n$*,* $u, u' \in I_p$*,* $\eta, \eta' \in \mathcal{P}(E_p)$ *and* $f \in Osc_1(E_n)$ *we have the following Lipschitz inequalities*

$$\left|\Gamma^1_{p,n}(u',\eta') - \Gamma^1_{p,n}(u,\eta)\right| \leq c^{1,1}_{p,n} \; |u - u'| + c^{1,2}_{p,n} \int \left|[\eta - \eta'](\varphi)\right| \; \Sigma^1_{p,n,u',\eta'}(d\varphi)$$

$$\left|\Gamma^2_{p,n}(u',\eta')(f) - \Gamma^2_{p,n}(u,\eta)(f)\right] \leq c^{2,1}_{p,n} \; |u - u'| + c^{2,2}_{p,n} \int \left|[\eta - \eta'](\varphi)\right| \; \Sigma^2_{p,n,u'\eta'}(f, d\varphi)$$

for some probability measures $\Sigma^1_{p,n,u',\eta'}(d\varphi)$ *and Markov transitions* $\Sigma^2_{p,n,m'\eta'}$*, with the collection of parameters*

$$c^{1,1}_{p,n} = a^1_{p,n} + \sum_{p \leq q < n} c^{2,1}_{p,q} \; \overline{a}^1_{q,n} \qquad \textit{and} \qquad c^{1,2}_{p,n} = \sum_{p \leq q < n} c^{2,2}_{p,q} \; \overline{a}^1_{q,n}$$

$$c^{2,1}_{p,n} = b'_{p,n} + \sum_{l=1}^{n-p} \sum_{p \leq r_1 < \ldots r_l < n} b'_{p,r_1} \prod_{1 \leq k \leq l} b_{r_k, r_{k+1}}$$

$$c^{2,2}_{p,n} = a^2_{p,n} + \sum_{l=1}^{n-p} \sum_{p \leq r_1 < \ldots r_l < n} a^2_{p,r_1} \prod_{1 \leq k \leq l} b_{r_k, r_{k+1}} \; , \quad \textit{with the convention } r_{l+1} = n.$$

In particular, the collection of parameters $\delta\left(\Sigma^i_{p,n}\right)_{i=1,2}$*,* $p \leq n$ *introduced in (1.16) and (2.12) are such that*

$$\delta\left(\Sigma^1_{p,n}\right) \leq c^{1,2}_{p,n} \quad \text{and} \quad \delta\left(\Sigma^2_{p,n}\right) \leq c^{2,2}_{p,n}$$

The proof of this proposition is rather technical and it is postponed to section 5.3 in the appendix. Now we conclude this section with a direct application of the above estimates. The proof of the theorem 1.1 stated in the introduction and the uniform estimates discussed in theorem 1.2 are a direct consequence of the following lemma.

Lemma 3.4 *Suppose that* $\tau^i = \sup_{n\geq 1} \tau^i_n < \infty$*, and* $a^i_{p,n} \leq c_i\ e^{-\lambda_i(n-p)}$*, for any* $p \leq n$*, and some finite parameters* $c_i < \infty$ *and* $\lambda_i > 0$*, with* $i = 1, 2$*, satisfying the following condition*

$$\lambda_1 \neq \lambda_2 \quad \text{and} \quad c_1 c_2\ \tau^1 \tau^2 \leq \left(1 - e^{-(\lambda_1\wedge\lambda_2)}\right)\ \left(e^{-(\lambda_1\wedge\lambda_2)} - e^{-(\lambda_1\vee\lambda_2)}\right)$$

Then, for any $i, j \in \{1, 2\}$ *we have*

$$c^{i,j}_{p,n} \leq c^{i,j}\ e^{-\lambda(n-p)} \quad \text{with} \quad \lambda = (\lambda_1 \wedge \lambda_2) - \log\left(1 + c\tau^1\tau^2 \frac{e^{(\lambda_1\wedge\lambda_2)}}{e^{-(\lambda_1\wedge\lambda_2)} - e^{-(\lambda_1\vee\lambda_2)}}\right) > 0$$

and the parameters $c^{i,j}$ *defined below*

$$\begin{array}{lllllll} c^{2,2} & = & c_2 & & c^{2,1} & = & c_1 c_2 \tau^2 / \left(e^{-(\lambda_1\wedge\lambda_2)} - e^{-(\lambda_1\vee\lambda_2)}\right) \\ c^{1,1} & = & c_1 & \left(1 + c^{2,1}\tau^1/(e^{-\lambda} - e^{-\lambda_1})\right) & c^{1,2} & = & c_1 c_2 \tau^1 / (e^{-\lambda} - e^{-\lambda_1}) \end{array}$$

In particular, for any N*-approximation models* $(\gamma^N_n(1), \eta^N_n)$ *of the flow* $(\gamma_n(1), \eta_n)$ *satisfying condition (1.6), the* $\mathbb{L}_r$*-mean error estimates presented in (1.16) are uniform w.r.t. the time parameter*

$$\sup_{n\geq 0} \mathbb{E}\left(\left|V^{\gamma,N}_n(1)\right|^r\right)^{\frac{1}{r}} \leq a_r\ c^{1,2}/(1 - e^{-\lambda}) \quad \textit{and} \quad \sup_{n\geq 0} \mathbb{E}\left(\left|V^{\eta,N}_n(f)\right|^r\right)^{\frac{1}{r}} \leq a_r\ c^{2,2}/(1 - e^{-\lambda})$$

with some constants $a_r < \infty$ *whose values only depend on* r*.*

Proof:
Under the premise of the lemma

$$b_{p,n} \leq c\tau \sum_{p<q<n} e^{-\lambda_1(q-(p+1))}\ e^{-\lambda_2(n-(q+1))} \quad \text{and} \quad b'_{p,n} \leq c\tau^2 \sum_{p\leq q<n} e^{-\lambda_1(q-p)}\ e^{-\lambda_2(n-(q+1))}$$

with $c = c_1 c_2$ and $\tau = \tau^1\tau^2$. We further assume that $\lambda_1 > \lambda_2$ and we set $\Delta = |\lambda_1 - \lambda_2|$.

$$b_{p,n} \leq c\tau e^{-\lambda_2((n-1)-(p+1))} \sum_{p<q<n} e^{-\Delta(q-(p+1))} \leq c\tau e^{-\lambda_2((n-1)-(p+1))}/(1 - e^{-\Delta})$$

In the same way, if $\lambda_2 > \lambda_1$ we have

$$b_{p,n} \leq c\tau e^{-\lambda_1((n-1)-(p+1))} \sum_{p<q<n} e^{-\Delta(n-(q+1))} \leq c\tau e^{-\lambda_1((n-1)-(p+1))}/(1 - e^{-\Delta})$$

This implies that

$$b_{p,n} \leq c\tau e^{-(\lambda_1\wedge\lambda_2)((n-1)-(p+1))}/(1-e^{-\Delta})$$

In much the same way, it can be shown that

$$b'_{p,n} =\leq c\tau^2 e^{-(\lambda_1\wedge\lambda_2)((n-1)-p)}/(1-e^{-\Delta}) \tag{3.7}$$

We are now in a position to estimate the parameters $c^{i,j}_{p,n}$. Firstly, we observe that

$$c^{2,2}_{p,n} \leq c_2\ e^{-\lambda_2(n-p)} + c_2 \sum_{l=1}^{n-p} \left(\frac{c\tau^1\tau^2 e^{2(\lambda_1\wedge\lambda_2)}}{1-e^{-\Delta}} \right)^l \sum_{p\leq r_1<\ldots r_l<n} e^{-\lambda_2(r_1-p)} e^{-(\lambda_1\wedge\lambda_2)(n-r_1)}$$

When $\lambda_1 > \lambda_2$, we find that

$$c^{2,2}_{p,n} \leq c_2\ e^{-\lambda_2(n-p)} \sum_{l=0}^{n-p} \left(\frac{c\tau e^{2\lambda_2}}{1-e^{-\Delta}} \right)^l \begin{pmatrix} n-p \\ l \end{pmatrix}$$

and therefore

$$c^{2,2}_{p,n} \leq c_2\ e^{-\lambda_2(n-p)} \left(1 + c\tau\ \frac{e^{2\lambda_2}}{1-e^{-\Delta}}\right)^{n-p} \Rightarrow c^{2,2}_{p,n} = c_2\ e^{-\lambda(n-p)}$$

with

$$\lambda = \lambda_2 - \log\left(1 + c\tau \frac{e^{\lambda_2}}{e^{-\lambda_2} - e^{-\lambda_1}}\right) > 0$$

as long as

$$c\tau \leq \left(1 - e^{-\lambda_2}\right)\ \left(e^{-\lambda_2} - e^{-\lambda_1}\right)$$

When $\lambda_2 > \lambda_1$ we have $\lambda_2 = \lambda_1 + \Delta$, we find that

$$c^{2,2}_{p,n} \leq c_2\ e^{-\lambda_2(n-p)} + c_2 e^{-\lambda_1(n-p)} \sum_{l=1}^{n-p} \left(\frac{c\tau e^{2\lambda_1}}{1-e^{-\Delta}} \right)^l \sum_{p\leq r_1<\ldots r_l<n} e^{-\Delta(r_1-p)}$$

from which it follows that

$$c^{2,2}_{p,n} \leq c_2\ e^{-\lambda_1(n-p)} \left(1 + c\tau \frac{e^{2\lambda_1}}{1-e^{-\Delta}}\right)^{n-p}$$

Using a similar line of argument as above, we have

$$c^{2,2}_{p,n} \leq c_2\ e^{-\lambda(n-p)}$$

with

$$\lambda = \lambda_1 - \log\left(1 + c\tau \frac{e^{\lambda_1}}{e^{-\lambda_1} - e^{-\lambda_2}}\right) > 0$$

as long as

$$c\tau \leq \left(1 - e^{-\lambda_1}\right)\ \left(e^{-\lambda_1} - e^{-\lambda_2}\right)$$

We conclude that

$$c^{2,2}_{p,n} \leq c_2\ e^{-\lambda(n-p)}$$

with

$$\lambda = (\lambda_1 \wedge \lambda_2) - \log\left(1 + c\tau \frac{e^{(\lambda_1\wedge\lambda_2)}}{e^{-(\lambda_1\wedge\lambda_2)} - e^{-(\lambda_1\vee\lambda_2)}}\right) > 0$$

as long as

$$c\tau \leq \left(1 - e^{-(\lambda_1\wedge\lambda_2)}\right)\ \left(e^{-(\lambda_1\wedge\lambda_2)} - e^{-(\lambda_1\vee\lambda_2)}\right)$$

Using (3.7) we also show that

$$c^{2,1}_{p,n} \leq c^{2,1}\ e^{-\lambda(n-p)} \quad \text{with} \quad c^{2,1} = c\tau^2\ \frac{1}{e^{-(\lambda_1\wedge\lambda_2)} - e^{-(\lambda_1\vee\lambda_2)}}$$

Using these estimates

$$c^{1,1}_{p,n} = c_1\ e^{-\lambda_1(n-p)} + \sum_{p\leq q<n} c^{2,1}_{p,q}\ c_1\tau^1\ e^{-\lambda_1(n-(q+1))}$$

and

$$c^{1,1}_{p,n} = c_1\ e^{-\lambda_1(n-p)} + c^{2,1}c_1\tau^1 \sum_{p\leq q<n} e^{-\lambda(q-p)}\ e^{-\lambda_1(n-(q+1))}$$

Since $\lambda_1 > \lambda$ we find that

$$c^{1,1}_{p,n} \leq c_1\ e^{-\lambda_1(n-p)} + c^{2,1}c_1\tau^1\ e^{-\lambda((n-1)-p)}/(1-e^{-\Delta'}) \quad \text{with} \quad \Delta' = \lambda_1 - \lambda > 0$$

This yields

$$c^{1,1}_{p,n} \leq c^{1,1}\ e^{-\lambda(n-p)} \quad \text{with} \quad c^{1,1} := c_1\ \left(1 + c^{2,1}\tau^1/(e^{-\lambda} - e^{-\lambda_1})\right)$$

Finally, we observe that

$$c^{1,2}_{p,n} = c\tau^1 \sum_{p\leq q<n} e^{-\lambda(q-p)}\ e^{-\lambda_1(n-(q+1))} \leq c\tau^1\ e^{-\lambda((n-1)-p)}/(1-e^{-\Delta'})$$

which implies that

$$c^{1,2}_{p,n} \leq c^{1,2}\ e^{-\lambda(n-p)} \quad \text{with} \quad c^{1,2} := c\tau^1/(e^{-\lambda} - e^{-\lambda_1})$$

This ends the proof of the lemma. ■

3.2 Feynman-Kac models

We let $Q_{p,n}$, with $0 \leq p \leq n$, be the Feynman-Kac semi-group associated with a sequence of bounded and positive integral operator Q_n from some measurable spaces $(E_{n-1}, \mathcal{E}_{n-1})$ into $(E_n, \mathcal{E}_n)$. For any $n \geq 1$, we denote by G_{n-1} and M_n the potential function on E_{n-1} and the Markov transition from E_{n-1} into E_n defined below

$$G_{n-1}(x) = Q_n(1)(x) \quad \text{and} \quad M_n(f)(x) = \frac{Q_n(f)(x)}{Q_n(1)(x)}$$

We also denote by $\Phi_{p,n}$, $0 \leq p \leq n$, the nonlinear semigroup from $\mathcal{P}(E_p)$ into $\mathcal{P}(E_n)$ defined below

$$\forall \eta \in \mathcal{P}(E_p),\ \forall f \in \mathcal{B}(E_n) \quad \Phi_{p,n}(\eta)(f) = \eta Q_{p,n}(f)/\eta Q_{p,n}(1) \tag{3.8}$$

As usual we use the convention $\Phi_{n,n} = Id$, for $p = n$. It is important to observe that this semigroup is alternatively defined by the formulae

$$\Phi_{p,n}(\eta)(f) = \frac{\eta(G_{p,n}\ P_{p,n}(f))}{\eta(G_{p,n})} \quad \text{with} \quad G_{p,n} = Q_{p,n}(1) \quad \text{and} \quad P_{p,n}(f_n) = Q_{p,n}(f_n)/Q_{p,n}(1)$$

The next two parameters

$$r_{p,n} = \sup_{x,x' \in E_p} \frac{G_{p,n}(x)}{G_{p,n}(x')} \quad \text{and} \quad \beta(P_{p,n}) = \sup_{x_p,y_p \in E_p} \|P_{p,n}(x_p,.) - P_{p,n}(y_p,.)\|_{\rm tv} \tag{3.9}$$

measure respectively the relative oscillations of the potential functions $G_{p,n}$ and the contraction properties of the Markov transition $P_{p,n}$. Various estimates in the forthcoming sections will be expressed in terms of these parameters. For instance and for further use in several places in this article, we have the following Lipschitz regularity property.

Proposition 3.5 ([6]) *For any $f_n \in Osc_1(E_n)$ we have*

$$|[\Phi_{p,n}(\eta_p) - \Phi_{p,n}(\mu_p)]\,(f_n)| \leq\ 2\ r_{p,n}\ \beta(P_{p,n}) \quad \left|[\eta_p - \mu_p]\,\overline{P}^{\mu_p}_{p,n}(f_n)\right| \tag{3.10}$$

for some function $\overline{P}^{\mu_p}_{p,n}(f_n) \in Osc_1(E_p)$ that doesn't depends on the measure η_p.

Our next objective is to estimate the the contraction coefficients $r_{p,n}$ and $\beta(P_{p,n})$ in terms of the mixing type properties of the semigroup

$$M_{p,n}(x_p, dx_n) := M_{p+1}M_{p+2}\ldots M_n(x_p, dx_n)$$

associated with the Markov operators M_n. We introduce the following regularity condition.

$(MG)_m$ *There exists an integer $m \geq 1$ and a sequence $(\epsilon_p(M))_{p\geq 0} \in (0,1)^{\mathbb{N}}$ and some finite constant r_p such that for any $p \geq 0$ and any $(x,x') \in E_p^2$ we have*

$$M_{p,p+m}(x_p,.) \geq \epsilon_p(m) \quad M_{p,p+m}(x'_p,.) \quad \text{and} \quad G_p(x) \leq r_p\ G_n(x') \tag{3.11}$$

It is well known that the above condition is satisfied for any aperiodic and irreducible Markov chains on finite spaces. Loosely speaking, for non compact spaces this condition is related to the tails of the transition distributions on the boundaries of the state space. For instance, let us suppose that $E_n = \mathbb{R}$ and M_n is the bi-Laplace transition given by

$$M_n(x, dy) = \frac{c(n)}{2}\ e^{-c(n)\,|y - A_n(x)|}\ dy$$

for some $c(n) > 0$ and some drift function A_n with bounded oscillations $\mathrm{osc}(A_n) < \infty$. In this case, it is readily checked that condition $(M)_m$ holds true for $m = 1$ with the parameter $\epsilon_{n-1}(1) = \exp\left(-c(n)\ \mathrm{osc}(A_n)\right)$.

Under the mixing type condition $(M)_m$ we have for any $n \geq m \geq 1$, and $p \geq 1$

$$r_{p,p+n} \leq \epsilon_p(m)^{-1} \prod_{0\leq k<m} r_{p+k} \tag{3.12}$$

and

$$\beta(P_{p,p+n}) \leq \prod_{k=0}^{\lfloor n/m\rfloor -1} \left(1-\epsilon^{(m)}_{p+km}\right) \quad \text{with} \quad \epsilon^{(m)}_p := \epsilon^2_p(m) \prod_{0<k<m} r^{-1}_{p+k} \tag{3.13}$$

Notice that these estimates are also valid for any $n \geq 0$. Several contraction inequalities can be deduced from these estimates (see for instance chapter 4 of the book [5]). To give a flavor of these results, we further assume that $(M)_m$ is satisfied with $m = 1$, and we have $\epsilon = \inf_n \epsilon_n(1) > 0$. In this case, we can show that

$$r_{p,p+n} \leq r_p/\epsilon \quad \text{and} \quad \beta(P_{p,p+n}) \leq \left(1-\epsilon^2\right)^n$$

We end this short section with a direct consequence of proposition 3.5.

Corollary 3.6 *Consider the Bernoulli semigroup presented in section 2.1. For constant mappings $s_n = \mu_{n+1}(1)$, the first component mapping is constant $\Phi^1_{n+1,\nu_n}(u) = s_n$ and the second component mapping $\Phi^2_{n+1,m_n}(\eta) = \Psi_{g^{(s)}_n}(\eta) M^{(s)}_{n+1}$ induces a Feynman-Kac semigroup with the likelihood function $g^{(s)}_n$ and the Markov transitions $M^{(s)}_{n+1}$ defined in (2.7). In this situation, the condition (3.2) is clearly met with $a^1_{p,n} = 0$, for any $p < n$,. We further assume that the semigroup of associated with the Markov transitions M_n satisfies the mixing property stated in the l.h.s. of (3.11) for some integer $m \geq 1$ and some parameter $\epsilon_p(m) \in]0,1]$. In this situation, the condition (3.3) is also met with the collection of parameters $a^2_{p,n}$ given below*

$$a^2_{p,n} \leq 2\ \rho_p(m) \prod_{k=0}^{\lfloor (n-p)/m\rfloor -1} \left(1-\epsilon^{(m,s)}_{p+km}\right)$$

with

$$\rho_p(m) := \epsilon^{-1}_p(m) \prod_{p\leq k<p+m} r^2_k(s_k) r_k(1) \quad \textit{and} \quad \epsilon^{(s,m)}_p = \epsilon^2_p(m) r_p(s_p) / \prod_{p\leq k<p+m} r_k(s_k)^3 r_k(1)^2$$

and the collection of parameters $r_n(s_n)$ defined below

$$r_n(s_n) := \frac{s_n g^+_n + (1-s_n)}{s_n g^-_n + (1-s_n)} \, (\leq r_n(1))$$

3.3 Bernoulli models

This section is concerned with the contraction properties of the semigroups $\Phi^1_{p,n,\nu}$ and $\Phi^2_{p,n,m}$ associated with the Bernoulli filter discussed in section 2.1. Before proceeding, we provide a brief discussion on the oscillations of the likelihood functions g_n given below

$$g_n(x_n) = (1-d_n(x_n)) + d_n(x_n) \mathcal{Y}_n\left(l_n(x_n, \cdot)/h_n\right)$$

in terms of some $[0,1]$-valued detection probability functions d_n, some local likelihood functions l_n, and some positive clutter intensity function h_n. The oscillations of these likelihood functions strongly depend on the nature of the functions (d_n, h_n, l_n).

Assuming that $h_n^- > 0$ we have

$$(1-d_n^{\circ,-}) + d_n^{\circ,-}\ \frac{l_n^-}{h_n^+}\ \mathcal{Y}_n(1) \leq g_n^- \leq g_n^+ \leq (1-d_n^{\circ,+}) + d_n^{\circ,+}\ \frac{l_n^+}{h_n^-}\ \mathcal{Y}_n(1) \tag{3.14}$$

with the parameters

$$d_n^{\circ,+} = d_n^+\ 1_{l_n^+\mathcal{Y}_n(1)\geq h_n^-} + d_n^-\ 1_{l_n^+\mathcal{Y}_n(1)<h_n^-}$$
$$d_n^{\circ,-} = d_n^-\ 1_{l_n^-\mathcal{Y}_n(1)\geq h_n^+} + d_n^+\ 1_{l_n^-\mathcal{Y}_n(1)<h_n^+}$$

The semigroup contraction inequalities developed in this section will be expressed in terms of the following parameters

$$\delta_n(sg) := \frac{g_n^+ s_n^+}{g_n^- s_n^-}\ , \quad \delta_n(g) := \frac{g_n^+}{g_n^-} \quad \text{and} \quad \delta_n'(g) := \frac{1}{g_n^-} \wedge g_n^+$$

For time homogeneous models $(d_n, h_n, l_n) = (d, h, l)$, with constant detection probability $d_n(x) = d$ and uniformly bounded number of observations $\sup_n \mathcal{Y}_n(1) \leq \mathcal{Y}^+(1) < \infty$ we have the following estimates

$$(1-d) \leq g_n^- \leq g_n^+ \leq (1-d) + d\ \frac{l^+}{h^-}\ \mathcal{Y}^+(1)$$

In this situation, we have

$$\delta_n(g) \leq 1 + \frac{d}{1-d}\ \frac{l^+}{h^-}\ \mathcal{Y}^+(1)$$

For small clutter intensity function with $h^- > 0$ and $l^- > 0$ we also have the observation free estimates $\frac{g_n^+}{g_n^-} \leq \frac{l^+h^+}{l^-h^-}$, from which we find that the upper bound

$$\delta(g) := \sup_{n\geq 0} \delta_n(g) \leq \inf\left\{1 + \frac{d}{1-d}\ \frac{l^+}{h^-}\ \mathcal{Y}(1)\,, \frac{l^+h^+}{l^-h^-}\right\} \tag{3.15}$$

and for $d < 1$

$$\delta'(g) := \sup_{n\geq 0} \delta_n'(g) \leq \sup\left\{(1-d) + d\frac{l^+}{h^-}\ \mathcal{Y}(1)\,, \frac{1}{1-d}\right\} \tag{3.16}$$

To be more precise, if we set $\inf_n \mathcal{Y}_n(1) = \mathcal{Y}^-(1)$ then

$$1 \leq \frac{l^-}{h^+}\mathcal{Y}(1)^- \Rightarrow \delta'(g) \leq (1-d) + d\ \frac{l^+}{h^-}\ \mathcal{Y}^+(1)$$

In addition, if we have $d(1-d)\mathcal{Y}(1) \leq h^-/l^+$ and $d < 1$ then we find the the observation free estimates

$$d\mathcal{Y}(1)\ l^+/h^- \leq 1/(1-d) \Rightarrow \delta'(g) \leq (1-d) + \frac{1}{1-d}$$

Conversely, we have the observation free estimates

$$\frac{l^+}{h^-}\mathcal{Y}(1)^+ \leq 1 \Rightarrow \delta'(g) \leq \frac{1}{(1-d) + d\ \frac{l^-}{h^+}\ \mathcal{Y}^-(1)} \leq \frac{1}{1-d}$$

We are now in position to state the main result of this section.

Theorem 3.7 *If $\mu_{n+1}(1) \in]0,1[$, $0 < s_n^- \leq s_n^+ < 1$, and the semigroup $M_{p,n}$ satisfies the condition stated in the l.h.s. of (3.11) for some integer $m \geq 1$ and some positive constant $\epsilon_p(m)$, then the condition $(Lip(\Phi))$ is met with*

$$a_{p,n}^1 \leq 2\ \epsilon_p^{-1}\delta_p'(g) \prod_{p\leq k<p+n} (1-\epsilon_k^2) \quad \text{and} \quad a_{p,n}^2 \leq 2\ \rho_p(m) \prod_{k=0}^{\lfloor n/m\rfloor -1} \left(1-\epsilon_{p+km}^{(m)}\right)$$

with some parameters

$$\epsilon_n \geq \inf\left\{\frac{s_n^-}{\mu_{n+1}(1)}, \frac{\mu_{n+1}(1)}{s_n^+}, \frac{1-s_n^+}{1-\mu_{n+1}(1)}, \frac{1-\mu_{n+1}(1)}{1-s_n^-}\right\}$$

and

$$\rho_p(m) \leq \epsilon_p(m)^{-1} \prod_{0\leq k<m} \delta_{p+k}(sg)^3 \quad \text{and} \quad \epsilon_p^{(m)} \geq \epsilon_p(m)^2\ \delta_p(sg)^{-4} \prod_{0<k<m} \delta_{p+k}(sg)^{-5}$$

In addition condition $(Cont(\Phi))$ is met with

$$\tau_{n+1}^1 \leq \delta_n(g)\ \left[(s_n^+ - s_n^-) + \|s_n - \mu_{n+1}(1)\|\right] \quad \text{and} \quad \tau_{n+1}^2 \leq \delta_n'(g)\ \sup\left\{\frac{\mu_{n+1}}{s_n^-}, \frac{s_n^+}{\mu_{n+1}(1)}\right\}$$

The proof of the theorem is postponed to section 5.2. To give a flavour of these estimates we examine time homogeneous models

$$(d_n, h_n, l_n, s_n, \mu_n) = (d, h, l, s, \mu)$$

with constant detection and survival probabilities $d_n(x) = d$, $s_n(x) = s$, and uniformly bounded number of observations $\sup_n \mathcal{Y}_n(1) \leq \mathcal{Y}(1) < \infty$. In this situation, we have $(\epsilon_p(m), \epsilon_p^{(s)}(m)) = (\epsilon(m), \epsilon^{(s)}(m))$ and using the estimates (3.15) we prove the following bounds

$$\tau_{n+1}^1 \leq \delta(g)\ |s-\mu(1)| \quad \text{and} \quad \tau_{n+1}^2 \leq \delta'(g)\ \frac{\mu(1)\vee s}{\mu(1)\wedge s}$$

and

$$a_{0,n}^1 \leq 2\epsilon^{-1}\delta'(g)\left(1-\epsilon^2\right)^n \quad \text{and} \quad a_{0,n}^2 \leq 2\epsilon(m)^{-1}\delta(g)^{3m}\ \left(1-\epsilon(m)^2\delta(g)^{-5m+1}\right)^{\lfloor n/m\rfloor}$$

with some parameter ϵ such that

$$\inf\left\{\frac{s}{\mu(1)}, \frac{\mu(1)}{s}, \frac{1-s}{1-\mu(1)}, \frac{1-\mu(1)}{1-s}\right\} \leq \epsilon \leq 1$$

It is also readily verified that the assumptions of lemma 3.4 are satisfied with the parameters

$$\begin{array}{rclrcl} \tau^1 & \leq & \delta(g)\ |s-\mu(1)| & \tau^2 & \leq & \delta'(g)\ ((\mu(1)\vee s)/(\mu(1)\wedge s)) \\ c_1 & = & 2\epsilon^{-1}\delta'(g) & c_2 & = & 2\epsilon(m)^{-1}\left(1-\epsilon(m)^2\delta(g)^{-5m+1}\right)^{-1}\delta(g)^{3m} \end{array}$$

and the Lyapunov constants

$$\lambda_1 = -\log\left(1-\epsilon^2\right) \quad \text{and} \quad \lambda_2 = -\frac{1}{m}\ \log\left(1-\epsilon(m)^2\delta(g)^{-5m+1}\right)$$

We notice that ϵ tends to 1 and τ^1 tends to 0, as $|s-\mu(1)|$ tends to 0. Thus, there exists some $\varsigma \geq 0$ such that

$$\lambda_1 > \lambda_2 \quad \text{and} \quad c_1 c_2 \tau^1 \tau^2 < \left(1-e^{-\lambda_2}\right)\ \left(e^{-\lambda_2}-e^{-\lambda_1}\right)$$

as long as $|s-\mu(1)| \leq \varsigma$. We summarize this discussion with the following corollary.

Corollary 3.8 *Consider the time homogeneous model discussed above. Under the assumptions of theorem 3.7, for any N-approximation models $(\gamma_n^N(1), \eta_n^N)$ of the Bernoulli model $(\gamma_n(1), \eta_n)$ satisfying condition (1.6), the $\mathbb{L}_r$-mean error estimates presented in (1.16) are uniform w.r.t. the time parameter*

$$\sup_{n\geq 0} \mathbb{E}\left(\left|V_n^{\gamma,N}(1)\right|^r\right)^{\frac{1}{r}} \leq a_r\ c^{1,2}/(1-e^{-\lambda}) \quad \textit{and} \quad \sup_{n\geq 0} \mathbb{E}\left(\left|V_n^{\eta,N}(f)\right|^r\right)^{\frac{1}{r}} \leq a_r\ c^{2,2}/(1-e^{-\lambda})$$

with the parameters $(c^{1,2}, c^{2,2}, \lambda)$ defined in lemma 3.4, and some finite constants $a_r < \infty$ whose values only depend on r.

Remark 3.9 *When $\mu_{n+1}(1) = 0$ we have seen in (2.8) that*

$$\Phi^1_{n+1,\nu_n}(u) = \Psi_{g_n}(\nu_n)(s_n) \times \theta_{\nu_n(g_n)}(u) \quad \text{and} \quad \Phi^2_{n+1,m_n}(\eta) = \Psi_{g_n s_n}(\eta) M_{n+1}$$

with the collection of mappings θ_a, with $a \in [0,\infty[$, defined in (2.9). Using the fact that

$$\left|\Phi^1_{n+1,\nu_n}(u) - \Phi^1_{n+1,\nu_n}(u')\right| \;=\; \frac{\Psi_{g_n}(\nu_n)(s_n)\ \nu_n(g_n)}{[\nu(g_n)u + (1-u)]\,[\nu(g_n)u' + (1-u')]}\ |u-u'|$$

one proves that (3.2) is met with the rather crude upper bound

$$a^1_{p,n} \leq \prod_{p\leq k<n} a_{k,k+1} \quad \text{and} \quad a^1_{k,k+1} \leq (s_k^+ g_k^+)/(1\wedge g_k^-)^2$$

We also notice that the second component mapping Φ^2_{n+1,m_n} doesn't depends on the parameter m_n, and it induces a Feynman-Kac semigroup of the same form as the one studied in section 3.2. Assuming that the mixing condition stated in the l.h.s. of (3.11) is satisfied some integer $m \geq 1$ and some parameter $\epsilon_p(m) > 0$, one can prove that (3.3) is met with the collection of parameters $a^2_{p,n}$ given below

$$a^2_{p,n} \leq 2\ \rho_p(m) \prod_{k=0}^{\lfloor (n-p)/m\rfloor -1} \left(1-\epsilon^{(m)}_{p+km}\right) \quad \textit{with} \quad \rho_p(m) = \epsilon_p^{-1}(m) \prod_{p\leq q<p+m} \delta_q(sg)$$

and the collection of parameters $\epsilon_p^{(m)} = \epsilon_p^{(m)} = \epsilon_p^2(m)/\prod_{p<q<p+m}\delta_q(sg)$.

3.4 PHD Models

This section is concerned with the contraction properties of the semigroups $\Phi^1_{p,n,\nu}$ and $\Phi^2_{p,n,m}$ associated with the PHD filter discussed in section 1.1.2 and in section 2.2.

The analysis of these nonlinear models is much more involved than the one of the Bernoulli models. We simplify the analysis and we further assume that the clutter intensity function, the detectability rate as well as the survival and the spawning rates introduced in section 2.2 are time homogeneous and constants functions, and we set

$$(b_n(x), h_n(x), s_n(x), r_n(x)) = (b, h, s, r)$$

To simplify the presentation, we also assume that the state spaces, the Markov transitions of the targets, the likelihood functions and the spontaneous birth measures are time homogeneous, that is we have that $E_n = E$, $E_n^Y = E^Y$, $M_n = M$, $g_n(x, y) = g(x, y)$ and $\mu_{n+1} = \mu$. Without further mention, we suppose that $r(1-d) < 1$, $\mu(1) > 0$, $r > 0$, and for any $y \in E^Y$ we have

$$0 \leq g^-(y) := \inf_{x \in E} g(x, y) \leq g^+(y) := \sup_{x \in E} g(x, y) < \infty$$

Given a mapping θ from E^Y into $\mathbb{R}$, we set $\mathcal{Y}^-(\theta) := \inf_n \mathcal{Y}_n(\theta)$ and $\mathcal{Y}^+(\theta) := \sup_n \mathcal{Y}_n(\theta)$.

We recall from (1.10) that the PDH filter is defined by the measure-valued equation

$$\gamma_{n+1} = \gamma_n Q_{n+1,\gamma_n}$$

with the integral operator

$$Q_{n+1,\gamma_n}(x_n, dx_{n+1}) = g_{n,\gamma_n}(x_n) M_{n+1}(x_n, dx_{n+1}) + \gamma_n(1)^{-1} \; \mu_{n+1}(dx_{n+1})$$

with the function g_{n,γ_n} defined below

$$g_{n,\gamma_n}(x) = r(1-d) + rd \int \mathcal{Y}_n(dy) \; \frac{g(x, y)}{h + d\gamma_n(g(., y))}$$

We also notice that the total mass process and the normalized distribution flow are given by the following equations

$$\begin{aligned}
\gamma_{n+1}(1) &= \Phi^1_{n+1,\eta_n}(\gamma_n(1)) \\
&= \gamma_n(1) \; r(1-d) + \int \mathcal{Y}_n(dy) \; w_{\gamma_n(1)}(\eta_n, y) + \mu(1) \\
\eta_{n+1}(1) &= \Phi^2_{n+1,\gamma_n(1)}(\eta_n) \\
&\propto \gamma_n(1) \; r(1-d) \; \eta_n M + \int \mathcal{Y}_n(dy) \; w_{\gamma_n(1)}(\eta_n, y) \; \Psi_{g(.,y)}(\eta_n) M + \mu(1) \; \overline{\mu}
\end{aligned}$$

with the probability measure $\overline{\mu}$ and weight functions w defined below

$$\overline{\mu}(dx) = \mu(dx)/\mu(1) \quad \text{and} \quad w_u(\eta, y) := r \left(1 - \frac{h}{h + du\eta(g(., y))}\right)$$

For null clutter parameter $h = 0$, we already observe that the total mass transformation Φ^1_{n+1,η_n} doesn't depend on the flow of probability measures η_n and it is simply given by

$$\Phi^1_{n+1,\eta_n}(\gamma_n(1)) = \gamma_n(1) \; r(1-d) + r \; \mathcal{Y}_n(1) + \mu(1)$$

In this particular situation, we have

$$\gamma_n^N(1) = \gamma_n(1) = (r(1-d))^n \gamma_0(1) + \sum_{0 \leq k < n} (r(1-d))^{n-1-k} (r \; \mathcal{Y}_k(1) + \mu(1))$$

Now, we easily show that the pair of conditions (3.2) and (3.4) are satisfied with the parameters $a^1_{p,n} = (r(1-d))^{n-p}$ and $\tau^1_n = 0$. In more general situations, the total mass process is not explicitly known. Some useful estimates are provided by the following lemma.

Lemma 3.10 *We assume that the number of observations is uniformly bounded; that is, we have that $\mathcal{Y}^+(1) < \infty$. In this situation, the total mass process $\gamma_n(1)$ and any approximation model $\gamma_n^N(1)$ given by the recursion (1.4) (with the initial condition $\gamma_0^N(1) = \gamma_0(1)$) take values in a sequence of compact sets $I_n \subset [m^-, m^+]$ with*

$$m^- := \frac{\mu(1)}{1 - r(1-d)} \left(1 + rd\ \mathcal{Y}^-\left(\frac{g^-}{h + d\mu(1)g^-}\right)\right) \quad \textit{and} \quad m^+ := \gamma_0(1) + \frac{r\mathcal{Y}^+(1) + \mu(1)}{1 - r(1-d)}$$

Proof:
Using the fact that $\gamma_n(1) \geq \mu(1)$ we prove that

$$r\left(1 - \frac{h}{h + d\mu(1)\ g^-(y)}\right) \leq w_{\gamma_n(1)}(\eta_n, y) \leq r$$

from which we conclude that

$$\gamma_n(1)\ r(1-d) + r\ \mathcal{Y}_{h,n}(1) + \mu(1) \leq \Phi^1_{n+1,\eta_n}(\gamma_n(1)) \leq \gamma_n(1)\ r(1-d) + r\ \mathcal{Y}_n(1) + \mu(1)$$

with the random measures

$$\mathcal{Y}_{h,n}(dy) := \mathcal{Y}_n(dy)\ \frac{d\mu(1)\ g^-(y)}{h + d\mu(1)\ g^-(y)}$$

For any sequence of probability measures $\nu := (\nu_n)_{n\geq 0} \in \mathcal{P}(E)^{\mathbb{N}}$, and any starting mass $u \in [0, \infty[$ one conclude that

$$(r(1-d))^n\, u + \frac{r\mathcal{Y}_h^-(1) + \mu(1)}{1 - r(1-d)} \leq \Phi^1_{0,n,\nu}(u) \leq (r(1-d))^n\, u + \frac{r\mathcal{Y}^+(1) + \mu(1)}{1 - r(1-d)}$$

This implies that $\gamma_n(1), \gamma_n^N(1) \in I_n \subset [m^-, m^+]$ with

$$m^- := \frac{r\mathcal{Y}_h^-(1) + \mu(1)}{1 - r(1-d)} = \frac{\mu(1)}{1 - r(1-d)} \left(1 + rd\ \mathcal{Y}^-\left(\frac{g^-}{h + d\mu(1)g^-}\right)\right)$$

The end of the proof of the lemma is now completed. ■

We are now in position to state the main result of this section.

Theorem 3.11 *We assume that the number of observations is uniformly bounded; that is, we have that $\mathcal{Y}^+(1) < \infty$. In this situation, the condition $(Lip(\Phi))$ is met with the Lipschitz constants $a^i_{p,n} \leq \prod_{p\leq k<n} a^i_{k,k+1}$, with $i = 1, 2$, and the sequence of parameters $\left(a^i_{n,n+1}\right)_{n\geq 0}$, $i = 1, 2$, defined below*

$$a^1_{n,n+1} \leq r(1-d) + rdh\ \mathcal{Y}_n\left(\frac{g^+}{[h + dm^-g^-]^2}\right)$$

and

$$a^2_{n,n+1} \leq m^+ \frac{\beta(M)\left[(1-d) + d\ \mathcal{Y}_n\left(\frac{g^+}{h+dm^+g^+}\ \frac{g^+}{g^-}\right)\right] + hd\mathcal{Y}_n\left(\frac{g^+ - g^-}{(h+dm^-g^-)^2}\right)}{(1-d)\ m^- + dm^-\mathcal{Y}_n\left(\frac{g^-}{h+dm^-g^-}\right) + \mu(1)/r}$$

In addition, condition $(Cont(\Phi))$ is met with the sequence of parameters

$$\tau^1_{n+1} \leq rdhm^+\ \mathcal{Y}_n\left(\frac{g^+ - g^-}{[h + dm^-g^-]^2}\right) \quad \tau^2_{n+1} \leq \frac{(1-d) + hd\ \mathcal{Y}_n\left(\frac{g^+}{(h+dm^-g^-)^2}\right)}{(1-d)\ m^- + dm^-\mathcal{Y}_n\left(\frac{g^-}{h+dm^-g^-}\right) + \mu(1)/r}$$

The proof of theorem 3.11 is postponed to section 5.4.

Corollary 3.12 *We assume that $\mathcal{Y}^+\left(g^+/g^-\right)$ and $\mathcal{Y}^+\left(g^+/(g^-)^2\right)<\infty$. In this situation, there exists some parameters $0<\kappa_0\leq 1$, $\kappa_1<\infty$, and $\kappa_2>0$ such that for any $d\geq\kappa_0$, $\mu(1)\geq\kappa_1$, and $h\leq\kappa_2$, the semigroups $\Phi^1_{p,n,\nu}$ and $\Phi^2_{p,n,m}$ satisfy the pair of conditions $(Lip(\Phi))$ and $(Cont(\Phi))$ with some parameters $(a^i_{p,n},\tau^i_n)_{i=1,2,p\leq n}$, satisfying the assumptions of lemma 3.4. In particular, for any N-approximation models $(\gamma^N_n(1),\eta^N_n)$ of the PHD equation $(\gamma_n(1),\eta_n)$ satisfying condition (1.6), the $\mathbb{L}_r$-mean error estimates presented in (1.16) are uniform w.r.t. the time parameter*

$$\sup_{n\geq 0}\mathbb{E}\left(\left|V^{\gamma,N}_n(1)\right|^r\right)^{\frac{1}{r}}\leq a_r\ c^{1,2}/(1-e^{-\lambda})\quad \textit{and}\quad \sup_{n\geq 0}\mathbb{E}\left(\left|V^{\eta,N}_n(f)\right|^r\right)^{\frac{1}{r}}\leq a_r\ c^{2,2}/(1-e^{-\lambda})$$

with the parameters $(c^{1,2},c^{2,2},\lambda)$ defined in lemma 3.4, and some finite constants $a_r<\infty$ whose values only depend on r.

Proof:
There is no loss of generality to assume that $r(1-d)<1/2\leq d$ and $\mu(1)\geq 1\geq h$. Recalling that $m^-\geq\mu(1)$, one readily proves that

$$\frac{m^+}{\mu(1)}=\frac{\gamma_0(1)}{\mu(1)}+\frac{1}{1-r(1-d)}\left(1+\frac{r}{\mu(1)}\ \mathcal{Y}^+(1)\right)\leq 2+\gamma_0(1)+2r\mathcal{Y}^+(1):=\rho$$

If we set $\delta(g):=\rho\vee\mathcal{Y}^+\left(\frac{g^+}{g^-}\right)\vee\mathcal{Y}^+\left(\frac{g^+}{(g^-)^2}\right)$, then we find the rather crude estimates

$$a^1_{n,n+1}/r\leq(1-d)+\frac{2h}{\mu(1)^2}\ \delta(g)\quad\text{and}\quad a^2_{n,n+1}/r\leq\left[\beta(M)(1-d)+\frac{2h+\beta(M)}{\mu(1)}\ \right]\delta(g)$$

as well as

$$\tau^1_{n+1}/r\leq\frac{2h}{\mu(1)}\ \delta(g)^2\quad\text{and}\quad\tau^2_{n+1}/r\leq\frac{1}{\mu(1)}\left[(1-d)+\frac{2h}{\mu(1)^2}\ \delta(g)\right]$$

from which we find that

$$\tau^1\tau^2\leq\frac{2hr^2}{\mu(1)^2}\left[(1-d)+\frac{2h}{\mu(1)^2}\ \delta(g)\right]\ \delta(g)^2 \tag{3.17}$$

Thus, there exists some $0<\kappa_0\leq 1$ and some $\kappa_1<\infty$ so that for any $d\geq\kappa_0$ and any $\mu(1)\geq\kappa_1$ we have

$$\begin{aligned}a^1_{n,n+1}&\leq r\left[(1-d)+\frac{2}{\mu(1)^2}\right]\ \delta(g):=e^{-\lambda_1}<1\\ a^2_{n,n+1}&\leq r\left[(1-d)+\frac{3}{\mu(1)}\ \right]\delta(g):=e^{-\lambda_2}<1\quad\text{with}\quad 0<\lambda_2<\lambda_1\end{aligned}$$

Finally, using (3.17) we find some $\kappa_2>0$ such that for any $h\leq\kappa_2$, we have that $\tau^1\tau^2\leq\left(1-e^{-\lambda_2}\right)\left(e^{-\lambda_2}-e^{-\lambda_1}\right)$. The end of the proof is now a direct consequence of lemma 3.4. This ends the proof of the corollary. ■

4 Stochastic particle approximations

4.1 Mean field interacting particle systems

4.1.1 Description of the models

The mean field type interacting particle system associated with the equation (1.2) relies on the fact that the one step mappings Γ^2_{n+1} can be rewritten in the following form

$$\Gamma^2_{n+1}(\gamma_n(1),\eta_n) = \eta_n K_{n+1,\gamma_n} \quad \text{with} \quad \gamma_n = \gamma_n(1) \times \eta_n \tag{4.1}$$

for some collection of Markov kernels $K_{n+1,\gamma}$ indexed by the time parameter n and the set of measures $\gamma \in \mathcal{M}_+(E_n)$. We mention that the choice of the Markov transitions $K_{n,\gamma}$ is not unique. In the literature on mean field particle models, $K_{n,\gamma}$ are called a choice of McKean transitions. Some McKean interpretation models of the Bernoulli and the PHD filter models (1.8) and (1.10) are discussed in section 2.2 (see for instance (2.10)) and in section 2.1 (see for instance 2.2)

These models provide a natural interpretation of the distribution laws η_n as the laws of a non linear Markov chain $\overline{X}_n$ whose elementary transitions $\overline{X}_n \rightsquigarrow \overline{X}_{n+1}$ depends on the distribution $\eta_n = \text{Law}(\overline{X}_n)$, as well as on the current mass process $\gamma_n(1)$. In contrast to traditional McKean model, the dependency on the mass process induce a dependency of all the flow of measures η_p, for $0 \leq p \leq n$. For a thorough description of these discrete generation and non linear McKean type models, we refer the reader to [5].

In further developments of the article, we always assume that the mappings

$$\left(m, x_n, (x^i)_{1\leq i\leq N}\right) \mapsto K_{n+1,m\sum_{j=1}^N \delta_{x^j}}(x_n, A_{n+1}) \text{ and } G_{n+1,m\sum_{j=1}^N \delta_{x^j}}(x_n)$$

are pointwise known, and of course measurable w.r.t. the corresponding product sigma fields, for any $n \geq 0$, $N \geq 1$, $A_{n+1} \in \mathcal{E}_{n+1}$, and any $x_n \in E_n$. In this situation, the mean field particle interpretation of this nonlinear measure-valued model is an E_n^N-valued Markov chain $\xi_n^{(N)} = \left(\xi_n^{(N,i)}\right)_{1\leq i\leq N}$, with elementary transitions defined as

$$\gamma^N_{n+1}(1) = \gamma^N_n(1)\ \eta^N_n(G_{n,\gamma^N_n}) \tag{4.2}$$

$$\mathbb{P}\left(\xi^{(N)}_{n+1} \in dx \;\middle|\; \mathcal{F}^{(N)}_n\right) = \prod_{i=1}^N K_{n+1,\gamma^N_n}(\xi^{(N,i)}_n, dx^i) \tag{4.3}$$

with the pair of occupation measures $\left(\gamma^N_n, \eta^N_n\right)$ defined below

$$\eta^N_n := \frac{1}{N}\sum_{j=1}^N \delta_{\xi^{(N,j)}_n} \quad \text{and} \quad \gamma^N_n(dx) := \gamma^N_n(1)\ \eta^N_n(dx)$$

In the above displayed formula, $\mathcal{F}^N_n$ stands for the σ-field generated by the random sequence $(\xi^{(N)}_p)_{0\leq p\leq n}$, and $dx = dx^1 \times \ldots \times dx^N$ stands for an infinitesimal neighborhood of a point $x = (x^1, \ldots, x^N) \in E^N_n$. The initial system $\xi^{(N)}_0$ consists of N independent and identically distributed random variables with common law η_0. As usual, to simplify the presentation, when there is no possible confusion we suppress the parameter N, so that we write ξ_n and ξ^i_n instead of $\xi^{(N)}_n$ and $\xi^{(N,i)}_n$.

4.1.2 Convergence analysis

The rationale behind the mean field particle model described in (4.3) is that η_{n+1}^N is the empirical measure associated with N independent variables with distributions $K_{n+1,\gamma_n^N}\left(\xi_n^i,dx\right)$, so as long as γ_n^N is a good approximation of γ_n then η_{n+1}^N should be a good approximation of η_{n+1}. Roughly speaking, this induction argument shows that η_n^N tends to η_n, as the population size N tends to infinity.

These stochastic particle algorithms can be thought of in various ways: From the physical view point, they can be seen as microscopic particle interpretations of physical nonlinear measure-valued equations. From the pure mathematical point of view, they can also be interpreted as natural stochastic linearizations of nonlinear evolution semigroups. From the probabilistic point of view, they can be interpreted as a interacting recycling acceptance-rejection sampling techniques. In this case, they can be seen as a sequential and interacting importance sampling technique.

By construction, the local fluctuation random fields $(W_n^N)_{n\geq 0}$ defined in (1.5) can be rewritten as follows

$$\eta_n^N=\eta_{n-1}^N K_{n,\gamma_{n-1}^N}+\frac{1}{\sqrt{N}}\;W_n^N$$

Using Khintchine's inequality, we can check that (1.6) is met for any $r\geq 1$ and any $f_n\in \mathrm{Osc}_1(E_n)$, with the collection of universal constants given below

$$a_{2r}^{2r}\leq (2r)!\;2^{-r}/r!\quad\text{and}\quad a_{2r+1}^{2r+1}\leq (2r+1)!\;2^{-r}/r!$$

We end this section with a brief discussion on the PHD equation presented in (1.10). This model combines in a single step the traditional updating and a prediction filtering transition. This combination allows us to reduce the fluctuations of the local sampling errors and their propagations w.r.t. the time parameter. Since these updating-prediction models are often used in the literature of multiple target tracking, we provide below a short summary. If we set

$$\widehat{g}_{n,\gamma}^c(.,y)=\begin{cases}(1-d_n) & \text{if}\quad y=c\\ \dfrac{d_n g_n(.,y_n)}{h_n(y)+\gamma(d_n g_n(.,y))} & \text{if}\quad y\neq c\end{cases}$$

then

$$\gamma_{n+1}=\widehat{\gamma}_n Q_{n+1}+\mu_{n+1}\quad\text{with}\quad Q_{n+1}(f):=r_n\;M_{n+1}(f)$$

with the updated measures defined below

$$\widehat{\gamma}_n(f):=\gamma_n(\widetilde{g}_{n,\gamma_n}^c f)\quad\text{with}\quad \widetilde{g}_{n,\gamma_n}^c=\int\mathcal{Y}_n^c(dy)\;\widehat{g}_{n,\gamma_n}^c(.,y)$$

Notice that

$$\widehat{\gamma}_n(1)=\gamma_n(\widetilde{g}_{n,\gamma_n}^c f)\quad\text{and}\quad\widehat{\eta}_n(dx):=\widehat{\gamma}_n(dx)/\widehat{\gamma}_n(1)=\Psi_{\widetilde{g}_{\gamma_n,n}^c}(\eta_n)(dx)$$

from which we find the recursive formulae

$$\left(\begin{array}{c}\gamma_n(1)\\ \eta_n\end{array}\right)\xrightarrow{\quad\text{updating}\quad}\left(\begin{array}{c}\widehat{\gamma}_n(1)\\ \widehat{\eta}_n\end{array}\right)\xrightarrow{\quad\text{prediction}\quad}\left(\begin{array}{c}\gamma_{n+1}(1)\\ \eta_{n+1}\end{array}\right)$$

with the prediction transition described below

$$\gamma_{n+1}(1) = \widehat{\gamma}_n(r_n) + \mu_{n+1}(1) \quad \text{and} \quad \eta_{n+1} = \Psi_{r_n}\left(\widehat{\eta}_n\right) M'_{n+1,\widehat{\gamma}_n}$$

In the above displayed formula, $M'_{n+1,\widehat{\gamma}_n}$ is the Markov transition defined by

$$M'_{n+1,\widehat{\gamma}_n}(x,.) = \alpha'_n(\widehat{\gamma}_n)\ M_{n+1}(x,.) + \left(1-\alpha'_n(\widehat{\gamma}_n)\right)\ \overline{\mu}_{n+1}$$

with the collection of $[0,1]$-valued parameters $\alpha'_n(\widehat{\gamma}_n) = \widehat{\gamma}_n(r_n)/(\widehat{\gamma}_n(r_n)+\mu_{n+1}(1))$. It should be clear that the updating and the prediction transitions can be approximated using a genetic type selection and mutation transition. Each of these sampling transitions introduces a separate local sampling fluctuation error. The stochastic analysis of the corresponding mean field particle interpretations can be developed using the same line of arguments as those used for the particle model discussed above.

4.2 Interacting particle association systems

4.2.1 Description of the models

We let $(\mathcal{A}_n)_{n\geq 0}$ be a sequence of finite sets equipped with some finite positive measures $(\nu_n)_{n\geq 0}$. We further assume that the initial distribution γ_0 and the integral operators Q_{n+1,γ_n} in (1.1) have the following form

$$\gamma_0 = \int \nu_0(da)\ \eta_0^{(a)} \quad \text{and} \quad Q_{n+1,\gamma_n} = \int \nu_{n+1}(da)\ Q^{(a)}_{n+1,\gamma_n}$$

In the above display $\eta_0^{(a)}$ stands for a collection of measures on E_0, indexed by the parameter $a\in\mathcal{A}_0$, and $Q^{(a)}_{n+1,\gamma_n}$ is a collection of integral operators indexed by the parameter $a\in\mathcal{A}_{n+1}$. In this situation, we observe that

$$\gamma_0(1) = \nu_0(1) \quad \text{and} \quad \eta_0 = \int A_0(da)\ \eta_0^{(a)} \quad \text{with} \quad A_0(da) := \nu_0(da)/\nu_0(1)$$

We also assume that the following property is met

$$G^{(a)}_{n,\gamma} := Q^{(a)}_{n+1,\gamma}(1) \propto G^{(a)}_n \quad \text{and} \quad Q^{(a)}_{n+1,\gamma}(f)/Q^{(a)}_{n+1,\gamma}(1) := M^{(a)}_{n+1}(f) \tag{4.4}$$

for some function $G^{(a)}_n$ on E_n, and some Markov transitions $M^{(a)}_{n+1}$ from E_n into E_{n+1} whose values do not depend on the measures γ. For clarity of presentation, sometimes we write $\Psi^{(a)}_{G_n}$ instead of $\Psi_{G^{(a)}_n}$.

Definition 4.1 *We consider the collection of probability measures $\eta_n^{(\mathrm{a}_n)}\in\mathcal{P}(E_n)$, indexed by sequences of parameters*

$$\mathrm{a}_n = (a_0,\ldots,a_n) \in \mathcal{A}_{[0,n]} := (\mathcal{A}_0\times\ldots\times\mathcal{A}_n)$$

and defined by the following equations

$$\eta_n^{(\mathrm{a}_n)} = \left(\Phi_n^{(a_n)}\circ\ldots\circ\Phi_1^{(a_1)}\right)\left(\eta_0^{(a_0)}\right) \tag{4.5}$$

with the mappings $\Phi^{(a)}_n : \mathcal{P}(E_{n-1})\rightarrow\mathcal{P}(E_n)$ indexed by $a\in\mathcal{A}_n$ and defined by the updating-prediction transformation

$$\Phi^{(a)}_n(\eta) = \Psi^{(a)}_{G_{n-1}}(\eta)\, M^{(a)}_n$$

We illustrate these abstract conditions in the context of the multiple target tracking equation presented in (1.10). In this situation, it is convenient to add a pair of virtual observation states c, c' to E_n^Y. Using this notation, the above conditions are satisfied with the finite sets $\mathcal{A}_{n+1}$ and their counting measures ν_{n+1} defined below

$$\mathcal{A}_{n+1} = \left\{Y_n^i, 1 \leq i \leq N_n^Y\right\} \cup \{c, c'\} \qquad \nu_{n+1} = \mathcal{Y}_n + \delta_c + \delta_{c'} \in \mathcal{P}(\mathcal{A}_{n+1})$$

Using (1.10) and (1.12), we check that (4.4) is met with the couple of potential functions and Markov transitions defined by

$$(G_n^{(y_n)}, M_{n+1}^{(y_n)}) = \begin{cases} (r_n d_n g_n(., y_n), M_{n+1}) & \text{for} \quad y_n \notin \{c, c'\} \\ (r_n(1-d_n), M_{n+1}) & \text{for} \quad y_n = c \\ \left(1, \overline{\mu}_{n+1}\right) & \text{for} \quad y_n = c' \end{cases}$$

In this case, we observe that

$$Q_{n+1,\gamma_n}^{(y_n)}(x_n, .) = G_{n,\gamma_n}^{(y_n)}(x_n)\ M_{n+1}^{(y_n)}(x_n, .)$$

with the potential function $G_{n,\gamma_n}^{(y_n)}$ defined below

$$G_{n,\gamma_n}^{(y_n)} / G_n^{(y_n)} = \begin{cases} \left[h_n(y_n) + \gamma_n(d_n g_n(., y_n))\right]^{-1} & \text{for} \quad y_n \notin \{c, c'\} \\ 1 & \text{for} \quad y_n = c \\ \mu_{n+1}(1)/\gamma_n(1) & \text{for} \quad y_n = c' \end{cases} \tag{4.6}$$

Under our assumptions, using (1.2), we have the following result.

Proposition 4.2 *The solution the equation (1.2) has the following form*

$$\eta_n = \int A_n(da)\ \eta_n^{(a)}$$

with a total mass process $\gamma_n(1)$ and the association measures $A_n \in \mathcal{P}(\mathcal{A}_{[0,n]})$ defined by the following recursive equations

$$\gamma_{n+1}(1) = \gamma_n(1)\ \eta_n(G_{n,\gamma_n}) \quad \text{and} \quad A_{n+1} = \Omega_{n+1}\left(\gamma_n(1), A_n\right)$$

With the mapping

$$\Omega_{n+1} \ : \ (m, A) \in \left(]0, \infty[\times \mathcal{P}(\mathcal{A}_{[0,n]})\right) \mapsto \Omega_{n+1}(m, A) \in \mathcal{P}(\mathcal{A}_{[0,n+1]})$$

defined by the following formula

$$\Omega_{n+1}\left(m, A\right)\left(d(a,b)\right) \propto A(da)\ \nu_{n+1}(db)\ \eta_n^{(a)}\left(G^{(b)}_{n, m\int A(da)\ \eta_n^{(a)}}\right) \tag{4.7}$$

Proof:
The proof of the above assertion is simply based on the fact that

$$\begin{aligned} \eta_{n+1} \propto \int \nu_{n+1}(db)\ \eta_n Q_{n+1,\gamma_n}^{(b)} &= \int A_n(da)\ \nu_{n+1}(db)\ \eta_n^{(a)} Q_{n+1,\gamma_n}^{(b)} \\ &= \int A_n(da)\ \nu_{n+1}(db)\ \eta_n^{(a)}\left(G_{n,\gamma_n}^{(b)}\right)\ \eta_{n+1}^{(a,b)} \end{aligned}$$

This clearly implies that

$$\Gamma_n^2\left(m, \int A(da)\ \eta_{n-1}^{(a)})\right) = \int \Omega_n\,(m, A)\,(d(a,b))\ \eta_n^{(a,b)}$$

This ends the proof of the proposition. ∎

By construction, we notice that for any discrete measure $A \in \mathcal{P}(\mathcal{A}_{[0,n-1]})$, and any collection of measures $\eta^{(a)} \in \mathcal{P}(E_{n-1})$, with $a \in \mathcal{A}_{[0,n-1]}$ we have the formula

$$\Gamma_n^2\left(m, \int A(da)\ \eta^{(a)})\right) = \int \Omega_n\,(m, A)\,(d(a,b))\ \ \Phi_n^{(b)}\left(\eta^{(a)}\right)$$

4.2.2 Particle approximation models

To get some feasible solution, we further assume that $\eta_n^{(a)}\left(G_{n,\gamma_n}^{(b)}\right)$ are explicitly known for any sequence of parameters $(a,b) \in \left(\mathcal{A}_{[0,n]} \times \mathcal{A}_{n+1}\right)$. This rather strong condition is satisfied for the multiple target tracking model discussed above as long as the quantities

$$\eta_n^{(a_0,y_0,\ldots,y_{n-1})}(r_n d_n g_n(.,y_n)) \quad \eta_n^{(a_0,y_0,\ldots,y_{n-1})}\,(r_n(1-d_n)) \quad \eta_n^{(a_0,y_0,\ldots,y_{n-1})}(d_n g_n(.,y_n))$$

are explicitly known. This condition is clearly met for linear gaussian target evolution and observation sensors as long as the survival and detection probabilities s_n and d_n are state independent, and spontaneous birth $\overline{\mu}_n$ and spawned targets branching rates b_n are Gaussian mixtures. In this situation, the collection of measures $\eta_n^{(a_0,y_0,\ldots,y_{n-1})}$ are gaussian distributions and the equation (4.5) coincides with the traditional updating-prediction transitions of the discrete generation Kalman-Bucy filter.

We let $A_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathrm{a}_0^i}$, be the empirical measure associated with N independent and identically distributed random variables $(\mathrm{a}_0^i)_{1\leq i\leq N}$ with common distribution A_0. By construction, we have

$$\eta_0^N := \int A_0^N(da)\ \eta_0^{(a)} = \eta_0 + \frac{1}{\sqrt{N}}\ W_0^N$$

with some local sampling random fields satisfying (1.6). We further assume that $\gamma_0(1)$ is known and we set $\gamma_0^N = \gamma_0(1)\ \eta_0^N$.

$$\gamma_1^N(1) = \gamma_0^N(1)\ \eta_0^N(G_{0,\gamma_0^N}) \quad \text{and} \quad \eta_1^N := \int A_1^N(da)\ \eta_1^{(a)}$$

with the occupation measure $A_1^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathrm{a}_1^i}$ associated with N conditionally independent and identically distributed random variables $\mathrm{a}_1^i := (a_{0,1}^i, a_{1,1}^i)$ with common law $\Omega_1\left(\gamma_0^N(1), A_0^N\right)$. By construction, we also have

$$\eta_1^N := \int \Omega_1\left(\gamma_0^N(1), A_0^N\right)(da)\ \eta_1^{(a)} + \frac{1}{\sqrt{N}}\ W_1^N = \Gamma_1^2\left(\gamma_0^N(1), \eta_0^N\right) + \frac{1}{\sqrt{N}}\ W_1^N$$

with some local sampling random fields satisfying (1.6). Iterating this procedure, we define by induction a sequence of N-particle approximation measures

$$\gamma_n^N(1) = \gamma_{n-1}^N(1)\ \eta_{n-1}^N(G_{n-1,\gamma_{n-1}^N}) \quad \text{and} \quad \eta_n^N := \int A_n^N(da)\ \eta_n^{(a)}$$

with the occupation measure $A_n^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathrm{a}_n^i}$ associated with N conditionally independent and identically distributed random variables $\mathrm{a}_n^i := (a_{0,n}^i, a_{1,n}^i, \ldots, a_{n,n}^i)$ with common law $\Omega_n\left(\gamma_{n-1}^N(1), A_{n-1}^N\right)$. Arguing as above, we find that

$$\eta_n^N = \int \Omega_n\left(\gamma_{n-1}^N(1), A_{n-1}^N\right)(da)\ \eta_n^{(a)} + \frac{1}{\sqrt{N}}\ W_n^N = \Gamma_n^2\left(\gamma_{n-1}^N(1), \eta_{n-1}^N\right) + \frac{1}{\sqrt{N}}\ W_n^N$$

with some local sampling random fields satisfying (1.6).

4.2.3 Convergence analysis

The main objective of this section is to show that N-particle occupation measures A_n^N converge in a sense to be given, as N tends to ∞, to the association probability measures A_n. To this end we observe that the one step mapping Ω_{n+1} introduced in (4.7) can be rewritten in the following form

$$\Omega_{n+1}(m, A)(F) = \frac{A\mathcal{Q}_{n+1,mA}(F)}{A\mathcal{Q}_{n+1,mA}(1)}$$

with the collection of integral operators $\mathcal{Q}_{n+1,mA}$ from $\mathcal{A}_{[0,n]}$ into $\mathcal{A}_{[0,n+1]}$ defined below

$$\mathcal{Q}_{n+1,B}(a, d(a',b)) := \delta_a(da')\ \nu_{n+1}(db)\ \eta_n^{(a')}\left(\mathcal{G}_{n,B}^{(b)}\right) \quad \text{where} \quad \mathcal{G}_{n,B}^{(b)} := G^{(b)}_{n,\int B(da)\ \eta_n^{(a)}}$$

with $B = mA$. In the above display $d(a', b) = da' \times db$ stands for an infinitesimal neighborhood of the point $(a', b) \in \mathcal{A}_{[0,n+1]}$, with $a = (a'_0, \ldots, a'_n) \in \mathcal{A}_{[0,n]}$ and $b \in \mathcal{A}_{n+1}$, and $a = (a_0, \ldots, a_n) \in \mathcal{A}_{[0,n]}$. It is important to point out that

$$B_n := \gamma_n(1) \times A_n \Longrightarrow B_{n+1} = B_n\mathcal{Q}_{n+1,B_n}$$

Notice that the flow of measures $(B_n)_{n\geq 0}$ satisfies the same type of equation as in (1.1), with the a total mass evolution of the same form as (1.3):

$$B_{n+1}(1) = B_n(1)\ A_n\left(\mathcal{G}_{n,B_n}\right) \quad \text{with} \quad \mathcal{G}_{n,mA} := \int \nu_{n+1}(db)\ \mathcal{G}_{n,mA}^{(b)}$$

$$\mathcal{Q}_{n+1,B_n}(F)(a) = \int \nu_{n+1}(db)\ \eta_n^{(a)}\left(\mathcal{G}_{n,B_n}^{(b)}\right)\ F(a,b)$$

$$\left[\mathcal{Q}_{n+1,B}(F) - \mathcal{Q}_{n+1,B'}(F)\right](a) = \int \nu_{n+1}(db) \quad \left[\eta_n^{(a)}\left(\mathcal{G}_{n,B}^{(b)}\right) - \eta_n^{(a)}\left(\mathcal{G}_{n,B'}^{(b)}\right)\right]\ F(a,b)$$

If we set $B = mA$ and $B' = m'A'$ then condition (H_2') is met as long as

$$\left|\eta_n^{(a)}\left(\mathcal{G}_{n,B}^{(b)}\right) - \eta_n^{(a)}\left(\mathcal{G}_{n,B'}^{(b)}\right)\right| \leq c(n)\ |m - m'| + \int |[A - A'](\varphi)|\ \Sigma_{n,B'}^{(b)}(d\varphi)$$

for some collection of bounded measures $\Sigma^{(b)}_{n,B'}$ on $\mathcal{B}(\mathcal{A}_n)$ such that $\int \operatorname{osc}(\varphi)\ \Sigma^{(b)}_{n,B'} \leq \delta\left(\Sigma^{(b)}_n\right)$, for some finite constant $\delta\left(\Sigma^{(b)}_n\right) < \infty$, whose values do dot depend on the parameters $(m,A) \in (I_n \times \mathcal{P}(\mathcal{A}_n))$. Under the assumptions (4.4), we have

$$\mathcal{G}^{(b)}_{n,B}(x) = \alpha^{(b)}_n(B)\ G^{(b)}_n(x)$$

for some collection of parameters $\alpha^{(b)}_n(B)$ satisfying

$$\left|\alpha^{(b)}_n(B) - \alpha^{(b)}_n(B')\right| \leq c(n)\ |m-m'| + \int |[A-A'](\varphi)|\ \Sigma^{(b)}_{n,B'}(d\varphi)$$

This condition is clearly satisfied for the PHD model discussed in (4.6), as long as the functions $h_n(y_n) + d_n g_n(.,y_n)$ are uniformly bounded from above and below.

For instance, for $b = y_n \notin \{c,c'\}$ we have

$$\alpha^{(b)}_n(B) = \left[h_n(b) + \int B(da)\ \eta^{(a)}_n(d_n g_n(.,b))\right]^{-1}$$

In this case, we can check that

$$\left|\alpha^{(b)}_n(B) - \alpha^{(b)}_n(B')\right| \leq c(n)\left|[B-B'](\varphi^{(b)}_n)\right| \quad \text{with} \quad \varphi^{(b)}_n(a) := \eta^{(a)}_n(d_n g_n(.,b))$$

In the same way, we show that the condition (H_1) is also met for the PHD model. This, by construction of A^N_n we find that

$$A^N_n = \Omega_n\left(\gamma^N_{n-1}(1), A^N_{n-1}\right) + \frac{1}{\sqrt{N}}\ \mathcal{W}^N_n$$

with some local sampling random fields satisfying (1.6). Notice that

$$\Omega_{n+1}\left(m,A\right) = \Psi_{\mathcal{H}_{n,mA}}\left(A\right)\mathcal{M}_{n+1,mA}(a,d(a',b))$$

with the collection of potential functions

$$\mathcal{H}_{n,mA}(a) \quad := \quad \mathcal{Q}_{n+1,mA}(1)(a) = \eta^{(a)}_n\left(\mathcal{G}_{n,mA}\right)$$

and the Markov transitions

$$\mathcal{M}_{n+1,mA}(a,d(a',b)) \quad := \quad \frac{\mathcal{Q}_{n+1,mA}(a,d(a',b))}{\mathcal{Q}_{n+1,mA}(1)(a)} = \delta_a(da')\ \frac{\nu_{n+1}(db)\ \eta^{(a')}_n\left(\mathcal{G}^{(b)}_{n,mA}\right)}{\int \nu_{n+1}(db')\ \eta^{(a')}_n\left(\mathcal{G}^{(b')}_{n,mA}\right)}$$

4.3 Mixed particle association models

We consider the association mapping

$$\Omega_{n+1}\ :\ (m,A,\eta) \in \left(]0,\infty[\times\mathcal{A}_{[0,n]} \times \mathcal{P}(E_n)^{\mathcal{A}_{[0,n]}}\right) \mapsto \Omega_{n+1}(m,A,\eta) \in \mathcal{P}(\mathcal{A}_{[0,n+1]})$$

defined for any $(m,A) \in \left(]0,\infty[\times\mathcal{A}_{[0,n]}\right)$ and any mapping $\eta : a \in \operatorname{Supp}(A) \mapsto \eta^{(a)} \in Pa(E_n)$ by

$$\Omega_{n+1}\left(m,A,\eta\right)\left(d(a,b)\right) \propto A(da)\ \nu_{n+1}(db)\ \eta^{(a)}\left(G^{(b)}_{n,m\int A(da)\ \eta^{(a)}}\right)$$

By construction, for any discrete measure $A \in \mathcal{P}(\mathcal{A}_{[0,n-1]})$, and any mapping $a \in \text{Supp}(A) \mapsto \eta^{(a)} \in \mathcal{P}(E_{n-1})$, we have the formula

$$\Gamma_n^2\left(m, \int A(da)\ \eta^{(a)})\right) = \int \Omega_n\left(m, A, \eta^{(\cdot)}\right)(d(a,b))\ \ \Phi_n^{(b)}\left(\eta^{(a)}\right)$$

We also mention that the updating-prediction transformation defined in (4.5)

$$\Phi_n^{(a)}(\eta) = \Psi_{G_{n-1}}^{(a)}(\eta)\, M_n^{(a)} = \eta\mathcal{K}_{n,\eta}^{(a)} \quad \text{with} \quad \mathcal{K}_{n,\eta}^{(a)} = \mathcal{S}_{n-1,\eta}^{(a)} M_n^{(a)} \tag{4.8}$$

In the above displayed formula $\mathcal{S}_{n,\eta}^{(a)}$ stands for some updating Markov transition from E_{n-1} into itself satisfying the compatibility condition $\eta\mathcal{S}_{n-1,\eta}^{(a)} = \Psi_{G_{n-1}}^{(a)}(\eta)$.

We let $A_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathrm{a}_0^i}$, be the empirical measure associated with N independent and identically distributed random variables $(\mathrm{a}_0^i)_{1\leq i\leq N}$ with common distribution A_0. For any $\mathrm{a} \in \mathcal{A}_0$, we let

$$\eta_0^N := \int A_0^N(d\mathrm{a})\ \eta_0^{(\mathrm{a},N')} \quad \text{and} \quad \eta_0^{(\mathrm{a},N')} = \frac{1}{N'}\sum_{i=1}^{N'} \delta_{\xi_0^{[\mathrm{a},j]}}$$

with the empirical measure $\eta_0^{(\mathrm{a},N')}$ associated with N' random variables $\xi_0^{[\mathrm{a}]} = \left(\xi_0^{[\mathrm{a},j]}\right)_{1\leq j\leq N'}$ with common law $\eta_0^{(\mathrm{a})}$. We further assume that $\gamma_0(1)$ is known and set

$$\gamma_0^N := \gamma_0(1)\ \eta_0^N \quad \text{and} \quad \gamma_1^N(1) := \gamma_0^N(1)\ \eta_0^N(G_{0,\gamma_0^N})$$

It is readily checked that the fluctuation random fields given below

$$\mathcal{W}_0^{(a,N')} = \sqrt{N'}\ \left(\eta_0^{(a,N')} - \eta_0^{(a)}\right)$$

satisfies (1.6), with $N = N'$, for any given $a \in \mathcal{A}_0$. Using the fact that

$$\int A_0^N(d\mathrm{a})\ \eta_0^{(\mathrm{a},N')} = \int A_0^N(d\mathrm{a})\ \eta_0^{(\mathrm{a})} + \frac{1}{\sqrt{N'}}\ \int A_0^N(d\mathrm{a})\ \mathcal{W}_0^{(a,N')}$$

we conclude that

$$\eta_0^N := \eta_0 + \frac{1}{\sqrt{N}}\ W_0^N$$

with some local sampling random fields W_0^N satisfying the same estimates as in (1.6) by replacing $1/\sqrt{N}$ by the sum $\left(1/\sqrt{N} + 1/\sqrt{N'}\right)$.

Using (4.8), for any $\mathrm{a}_1 = (a_0, a_1)$ we find that

$$\Phi_1^{(a_1)}\left(\eta_0^{(a_0,N')}\right) = \eta_0^{(a_0,N')}\mathcal{K}_{n,\eta_0^{(a_0,N')}}^{(a_1)}$$

We let $A_1^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathrm{a}_1^i}$ be the occupation measure associated with N conditionally independent and identically distributed random variables $\mathrm{a}_1^i := (a_{0,1}^i, a_{1,1}^i)$ with common law

$$\Omega_1\left(\gamma_0^N(1), A_0^N, \eta_0^{(\cdot,N')}\right)$$

In the above displayed formula $\eta_0^{(\cdot,N')}$ stands for the mapping $a_0 \in \mathcal{A}_0 \mapsto \eta_0^{(a_0,N')} \in \mathcal{P}(E_0)$.

We consider a sequence of conditionally independent random variables $\xi_1^{[a_0,a_1,j]}$ with distribution $\mathcal{K}^{(a_1)}_{n,\eta_0^{(a_0,N')}}\left(\xi_0^{[a_0,j]},\cdot\right)$, with $1 \leq j \leq N'$, and we set

$$\eta_1^{((a_0,a_1),N')} = \frac{1}{N'} \sum_{i=1}^{N'} \delta_{\xi_1^{[(a_0,a_1),j]}} \quad \text{and} \quad \eta_1^N := \int A_1^N(d\mathrm{a})\ \eta_1^{(\mathrm{a},N')}$$

Arguing as before, for any given $\mathrm{a}_1 := (a_0,a_1) \in \mathrm{Supp}(A_1^N)$, the sequence of random fields

$$\mathcal{W}_1^{(\mathrm{a}_1 N')} := \sqrt{N}\ \left(\eta_1^{((a_0,a_1),N')} - \Phi_1^{(a_1)}\left(\eta_0^{(a_0,N')}\right)\right)$$

satisfies (1.6), with $N = N'$. Thus, we conclude that

$$\begin{aligned}\eta_1^N &= \int \Omega_1\left(\gamma_0^N(1), A_0^N, \eta_0^{(\cdot,N')}\right)(d(a_0,a_1))\ \Phi_1^{(a_1)}\left(\eta_0^{(a_0,N')}\right) + \frac{1}{\sqrt{N}}\ W_1^N \\ &= \Gamma_1^2\left(\gamma_0^N(1),\eta_0^N\right) + \frac{1}{\sqrt{N}}\ W_1^N\end{aligned}$$

with some local sampling random fields W_1^N satisfying the same estimates as in (1.6) by replacing $1/\sqrt{N}$ by the sum $\left(1/\sqrt{N}+1/\sqrt{N'}\right)$. Iterating this procedure, we define by induction a sequence of N-particle approximation measures

$$\gamma_n^N(1) = \gamma_{n-1}^N(1)\ \eta_{n-1}^N(G_{n-1,\gamma_{n-1}^N}) \quad \text{and} \quad \eta_n^N := \int A_n^N(da)\ \eta_n^{(a,N')}$$

with the occupation measure $A_n^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathrm{a}_n^i}$ associated with N conditionally independent and identically distributed random variables $\mathrm{a}_n^i := (a_{0,n}^i, a_{1,n}^i, \ldots, a_{n,n}^i)$ with common law $\Omega_n\left(\gamma_{n-1}^N(1), A_{n-1}^N, \eta_{n-1}^{(\cdot,N')}\right)$. Arguing as above, we find that

$$\eta_n^N = \int \Omega_n\left(\gamma_{n-1}^N(1), A_{n-1}^N, \eta_{n-1}^{(\cdot,N')}\right)(d(a,b))\ \Phi_n^{(b)}\left(\eta_{n-1}^{(a,N')}\right) = \Gamma_n^2\left(\gamma_{n-1}^N(1),\eta_{n-1}^N\right) + \frac{1}{\sqrt{N}}\ W_n^N$$

with some local sampling random fields satisfying the same estimates as in (1.6) by replacing $1/\sqrt{N}$ by the sum $\left(1/\sqrt{N}+1/\sqrt{N'}\right)$. As before, the N-particle occupation measures A_n^N converge as N tends to ∞ to the association probability measures A_n.

5 Appendix

5.1 Proof of corollary 3.6

For constant mappings $s_n = \mu_{n+1}(1)$, the mappings Φ^1_{n+1,ν_n} and Φ^2_{n+1,m_n} are given by

$$\Phi^1_{n+1,\nu_n}(u) = s_n \quad \text{and} \quad \Phi^2_{n+1,m_n}(\eta) = \Psi_{g_n^{(s)}}(\eta) M^{(s)}_{n+1}$$

with the likelihood function $g_n^{(s)}$ and the Markov transitions $M_{n+1}^{(s)}$ defined in (2.7). Firstly, we observe that $r_n(s_n) := \sup_{x,x' \in E_n} g_n^{(s)}(x)/g_n^{(s)}(x')$. We also notice that the second component mapping Φ^2_{n+1,m_n} does not depends on the parameter m_n, and it induces a Feynman-Kac semigroup of the same form as the one discussed in section 3.2.

Under the premise of the proposition, the semigroup of associated with the Markov transitions M_n satisfies the mixing property stated in the l.h.s. of (3.11) for some integer $m \geq 1$ and some parameter $\epsilon_p(m) \in]0,1]$. In this situation, we also have that

$$M^{(s)}_{p,p+m}(x,.) \geq \epsilon^{(s)}_p(m) \ M^{(s)}_{p,p+m}(x',.)$$

with some positive parameter

$$\epsilon^{(s)}_p(m) \geq \epsilon_p(m) / \prod_{p \leq k < p+m} r_k(s_k) r_k(1) \quad \text{and} \quad r_n(s_n) := \frac{s_n g_n^+ + (1-s_n)}{s_n g_n^- + (1-s_n)} \ (\leq r_n(1))$$

To prove this claim, firstly we observe that $M^{(s)}_{p,p+m}(x,.) \ll M^{(s)-}_{p,p+m}(x,.)$ and

$$\prod_{p \leq k < p+m} r_k(s_k)^{-1} \leq dM^{(s)}_{p,p+m}(x,.)/dM^{(s)-}_{p,p+m}(x,.) \leq \prod_{p \leq k < p+m} r_k(1)$$

with the semigroup $M^{(s)-}_{p,n}$ associated with the Markov transition

$$M^{(s)-}_{p,p+1}(x,.) = \alpha_{p+1} \ M_{p+1}(x,.) + (1-\alpha_{k+1}) \ \overline{\mu}_{k+1} \quad \text{with} \quad \alpha_{p+1} := \frac{s_k g_k^-}{s_k g_k^- + (1-s_k)}$$

Using the geometric representation

$$M^{(s)-}_{p,n}(x,.) = \left(\prod_{p<k\leq n} \alpha_k \right) M_{p,n}(x,.) + \sum_{p<k\leq n} (1-\alpha_k) \left(\prod_{k<l\leq n} \alpha_l \right) \overline{\mu}_k M_{k,n}$$

it can be verified that

$$M^{(s)-}_{p,p+m}(x,.) \geq \epsilon_p(m) \ M^{(s)-}_{p,p+m}(x',.) \geq \epsilon_p(m) \left(\prod_{p \leq k < p+m} g_k^-/g_k^+ \right) \ M^{(s)}_{p,p+m}(x',.)$$

from which we conclude that

$$M^{(s)}_{p,p+m}(x,.) \geq \epsilon^{(s)}_p(m) \ M^{(s)}_{p,p+m}(x',.) \quad \text{with} \quad \epsilon^{(s)}_p(m) \geq \epsilon_p(m) / \prod_{p \leq k < p+m} r_k(s_k) r_k(1)$$

We end the proof of the proposition combing the proposition 3.5 with the couple of estimates presented in (3.12) and (3.13). This ends the proof of the corollary. ∎

5.2 Proof of theorem 3.7

The formulae presented in (2.6) can be rewritten in terms of matrix operations as follows

$$[\gamma_{n+1}(1)\ ,\ 1-\gamma_{n+1}(1)] = [\widehat{\gamma}_n(1)\ ,\ 1-\widehat{\gamma}_n(1)] \left[\begin{array}{cc} \Psi_{g_n}(\eta_n)(s_n) & 1-\Psi_{g_n}(\eta_n)(s_n) \\ \mu_{n+1}(1) & 1-\mu_{n+1}(1) \end{array}\right]$$

and

$$[\widehat{\gamma}_n(1)\ ,\ 1-\widehat{\gamma}_n(1)] = \frac{[\gamma_n(1)\ ,\ 1-\gamma_n(1)] \left[\begin{array}{cc} \eta_n(g_n) & 0 \\ 0 & 1 \end{array}\right]}{[\gamma_n(1)\ ,\ 1-\gamma_n(1)] \left[\begin{array}{cc} \eta_n(g_n) & 0 \\ 0 & 1 \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \end{array}\right]}$$

With a slight abuse of notation, we set

$$\vartheta_n := [\gamma_n(1)\ ,\ 1-\gamma_n(1)] \quad \widehat{\vartheta}_n := [\widehat{\gamma}_n(1)\ ,\ 1-\widehat{\gamma}_n(1)] \quad \text{and} \quad 1 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

We also denote by $\mathcal{M}_{n+1,\eta_n}$ and $\mathcal{D}_{n,\eta_n}$ the stochastic and the diagonal matrices defined by

$$\mathcal{M}_{n+1,\eta_n} := \left[\begin{array}{cc} \Psi_{g_n}(\eta_n)(s_n) & 1-\Psi_{g_n}(\eta_n)(s_n) \\ \mu_{n+1}(1) & 1-\mu_{n+1}(1) \end{array}\right] \quad \text{and} \quad \mathcal{D}_{n,\eta_n} := \left[\begin{array}{cc} \eta_n(g_n) & 0 \\ 0 & 1 \end{array}\right] \tag{5.1}$$

In this notation, the above recursion can be rewritten in a more compact form

$$\vartheta_{n+1} = \widehat{\vartheta}_n\ \mathcal{M}_{n+1,\eta_n} \quad \text{and} \quad \widehat{\vartheta}_n = \frac{\vartheta_n\ \mathcal{D}_{n,\eta_n}}{\vartheta_n \mathcal{D}_{n,\eta_n} 1} \implies \vartheta_{n+1} = \frac{\vartheta_n\ \mathcal{Q}_{n+1,\eta_n}}{\vartheta_n \mathcal{Q}_{n+1,\eta_n} 1}$$

with the product of matrices $\mathcal{Q}_{n+1,\eta_n} = \mathcal{D}_{n,\eta_n}\mathcal{M}_{n+1,\eta_n}$.

$$\forall u \in I_p(\subset [0,1]) \qquad \left[\Phi^1_{p,n,\nu}(u), 1-\Phi^1_{p,n,\nu}(u)\right] = \frac{[u, 1-u]\ \mathcal{Q}_{p,n,\nu}}{[u, 1-u]\ \mathcal{Q}_{p,n,\nu}(1)}$$

with the matrix semigroup

$$\mathcal{Q}_{p,n,\nu} = \mathcal{Q}_{p+1,\nu_p}\mathcal{Q}_{p+2,\nu_{p+1}} \ldots \mathcal{Q}_{n,\nu_{n-1}}$$

These semigroups are again of the same form as the Feynman-Kac models discussed in section 3.2 with a two point state space. When $\mu_{n+1}(1) \in]0,1[$ and $0 < s_n^- \leq s_n^+ < 1$, we have for any $n \geq 0$ and any $i, i', j \in \{1,2\}$

$$\mathcal{M}_{n+1,\nu_n}(i,j) \geq \epsilon_n\ \mathcal{M}_{n+1,\nu_n}(i',j) \quad \text{and} \quad \sup_{i,i' \in \{1,2\}} \frac{\mathcal{Q}_{n+1,\nu_n}(1)(i)}{\mathcal{Q}_{n+1,\nu_n}(1)(i')} \leq \delta'_n(g)$$

The first assertion is a direct consequence of the proposition 3.5 with the couple of estimates presented in (3.12) and (3.13).

Using (2.3), we find that Φ^2_{n+1,m_n} induces a Feynman-Kac models of the same form as the one discussed in section 3.2. More precisely, we have that

$$\Phi^2_{n+1,m_n}(\eta) = \Psi_{\widehat{G}_{n,m_n}}(\eta)\widehat{M}_{n+1,m_n}$$

with the potential functions G_{n,m_n} and the Markov transitions $\widehat{M}_{n+1,m_n}$ defined in (2.4) and (2.5). Notice that

$$\sup_{x,x'\in E_n} \frac{\widehat{G}_{n,m_n}(x)}{\widehat{G}_{n,m_n}(x')} \leq \delta_n(sg)$$

and for any $x \in E_n$ and any $n \geq 0$

$$\delta_n(sg)^{-1}\ \widehat{M}^-_{n+1,m_n}(x,.) \leq \widehat{M}_{n+1,m_n}(x,.) \leq \delta_n(sg)\ \widehat{M}^-_{n+1,m_n}(x,.)$$

with the Markov transitions $\widehat{M}^-_{n+1,m_n}$ defined as $\widehat{M}^-_{n+1,m_n}$ by replacing the functions (s_n, g_n) by their lower bounds (s^-_n, g^-_n). To prove this claim, we use the fact that for any positive function f we have

$$\frac{d\widehat{M}_{n+1,m_n}(f)}{d\widehat{M}^-_{n+1,m_n}(f)} = \frac{m_n g^-_n s^-_n + (1-m_n)\mu_{n+1}(1)}{m_n g_n s_n + (1-m_n)\mu_{n+1}(1)} \times \frac{m_n g_n s_n M_{n+1}(f) + (1-m_n)\mu_{n+1}(1)\overline{\mu}_{n+1}(f)}{m_n g^-_n s^-_n + (1-m_n)\mu_{n+1}(1)\overline{\mu}_{n+1}(f)}$$

and the two series of inequalities

$$\delta_n(sg)^{-1} \leq \frac{m_n g^-_n s^-_n + (1-m_n)\mu_{n+1}(1)}{m_n g_n s_n + (1-m_n)\mu_{n+1}(1)} \leq 1$$

and

$$1 \leq \frac{m_n g_n s_n M_{n+1}(f) + (1-m_n)\mu_{n+1}(1)\overline{\mu}_{n+1}(f)}{m_n g^-_n s^-_n + (1-m_n)\mu_{n+1}(1)\overline{\mu}_{n+1}(f)} \leq \delta_n(sg)$$

With a slight abuse of notation, we write $\widehat{M}_{p,n}$, and respectively $\widehat{M}^-_{p,n}$, the semigroup associated with the Markov transitions $\widehat{M}_{n+1,m_n}$, and resp. $\widehat{M}^-_{n+1,m_n}$. Using the same argument as in the proof of corollary 3.6 it follows that

$$\widehat{M}^-_{p,p+m}(x,.) \geq \epsilon_p(m)\ \widehat{M}^-_{p,p+m}(x',.)$$

from which we conclude that

$$\widehat{M}_{p,p+m}(x,.) \geq \widehat{\epsilon}_p(m)\ \widehat{M}_{p,p+m}(x',.) \quad \text{with} \quad \widehat{\epsilon}_p(m) \geq \epsilon_p(m) \prod_{0\leq k<m} \delta_{p+k}(sg)^{-2}$$

using proposition 3.5 with the couple of estimates presented in (3.12) and (3.13), we check that (3.3) is satisfied with

$$a^2_{p,n} \leq 2\ \rho_p(m) \prod_{k=0}^{\lfloor n/m\rfloor-1} \left(1-\epsilon^{(m)}_{p+km}\right)$$

and some parameters

$$\epsilon^{(m)}_p := \widehat{\epsilon}_p(m)^2 \prod_{0<k<m} \delta_{p+k}(sg)^{-1} \geq \epsilon_p(m)^2\ \delta_p(sg)^{-4} \prod_{0<k<m} \delta_{p+k}(sg)^{-5}$$

and

$$\rho_p(m) := \widehat{\epsilon}_p(m)^{-1} \prod_{0\leq k<m} \delta_{p+k}(sg) \leq \epsilon_p(m)^{-1} \prod_{0\leq k<m} \delta_{p+k}(sg)^3$$

This ends the proof of the first assertion of the theorem. Next, we discuss condition (Cont(Φ)). We observe that

$$\Phi^1_{n+1,\nu}(u) = \frac{u\ \nu(g_n s_n) + (1-u)\mu_{n+1}(1)}{u\ \nu(g_n) + (1-u)}$$

After some manipulations

$$\begin{aligned}
&\Phi^1_{n+1,\nu}(u) - \Phi^1_{n+1,\nu'}(u) \\
&= \tfrac{u\nu'(g_n)}{u\nu'(g_n)+(1-u)}\ \left[\Psi_{g_n}(\nu) - \Psi_{g_n}(\nu')\right](s_n) \\
&\quad + \tfrac{u}{u\nu(g_n)+(1-u)} \tfrac{(1-u)}{u\nu'(g_n)+(1-u)}\ \left[\Psi_{g_n}(\nu)(s_n) - \mu_{n+1}(1)\right]\ \left[\nu-\nu'\right](g_n)
\end{aligned}$$

Recalling that the mapping $\theta_a(x) = ax/(ax+(1-x))$ in increasing on $[0,1]$ and using the fact that

$$\Psi_{g_n}(\nu) = \nu S_{n,\nu} \Longrightarrow \Psi_{g_n}(\nu) - \Psi_{g_n}(\nu') = \frac{g_n^+}{\nu(g_n)}\ (\nu-\nu')S_{n,\nu'}$$

with the Markov transition

$$S_{n,\nu'}(x,dx') = \frac{g_n(x)}{g_n^+(x)}\ \delta_x(dx') + \left(1 - \frac{g_n(x)}{g_n^+(x)}\right)\ \Psi_{g_n}(\nu')(dx')$$

we prove

$$\left|\Psi_{g_n}(\nu)(s_n) - \Psi_{g_n}(\nu')(s_n)\right| \le \frac{g_n^+}{g_n^-}\ \left|(\nu-\nu')S_{n,\nu'}(s_n)\right| \tag{5.2}$$

and for any $u \in I_n = [m_n^-, m_n^+]$

$$\begin{aligned}
&\left|\Phi^1_{n+1,\nu}(u) - \Phi^1_{n+1,\nu'}(u)\right| \\
&= \tfrac{m_n^+ g_n^+}{m_n^+ g_n^+ + (1-m_n^+)}\ \tfrac{g_n^+}{g_n^-}\ \left|(\nu-\nu')S_{n,\nu'}(s_n)\right| \\
&\quad + \tfrac{m_n^+ g_n^+}{m_n^+ g_n^+ + (1-m_n^+)} \tfrac{(1-m_n^-)}{m_n^- g_n^- + (1-m_n^-)}\ \|s_n - \mu_{n+1}(1)\|\ \left|\left[\nu-\nu'\right](g_n/g_n^-)\right|
\end{aligned}$$

This implies that

$$\begin{aligned}
\tau^1_{n+1} &\le \frac{m_n^+ g_n^+}{m_n^+ g_n^+ + (1-m_n^+)}\ \frac{g_n^+}{g_n^-}\ (s_n^+ - s_n^-) \\
&\quad + \frac{m_n^+ g_n^+}{m_n^+ g_n^+ + (1-m_n^+)}\ \frac{(1-m_n^-)}{m_n^- g_n^- + (1-m_n^-)}\ \|s_n - \mu_{n+1}\|\ \left(\frac{g_n^+}{g_n^-} - 1\right) \\
&\le \frac{g_n^+}{g_n^-}\ \left[(s_n^+ - s_n^-) + \|s_n - \mu_{n+1}(1)\|\right]
\end{aligned}$$

Using (2.3) we also find that

$$\Phi^2_{n+1,m}(\eta)(f) = \frac{m\eta(s_n g_n M_{n+1}(f)) + (1-m)\ \mu_{n+1}(f)}{m\eta(s_n g_n) + (1-m)\mu_{n+1}(1)}$$

It is also readily check that

$$\left[\Phi^2_{n+1,m}(\eta)-\Phi^2_{n+1,m'}(\eta)\right](f)=\frac{\mu_{n+1}(1)\ \eta(g_ns_n)\ \left[\Psi_{g_ns_n}(\eta)M_{n+1}-\overline{\mu}_{n+1}\right](f)\ (m-m')}{[m\eta(s_ng_n)+(1-m)\mu_{n+1}(1)]\,[m'\eta(s_ng_n)+(1-m')\mu_{n+1}(1)]}$$

from which we conclude that

$$\tau^2_{n+1}\leq\sup\left\{\frac{\mu_{n+1}}{s_n^-g_n^-},\frac{s_n^+g_n^+}{\mu_{n+1}(1)}\right\}\leq\delta'_n(g)\ \sup\left\{\frac{\mu_{n+1}}{s_n^-},\frac{s_n^+}{\mu_{n+1}(1)}\right\}$$

This ends the proof of the theorem. ∎

5.3 Proof of proposition 3.3

The proof of proposition 3.3 is based on the following technical lemma.

Lemma 5.1 *We assume that the regularity conditions* $(Lip(\Phi))$ *and* $(Cont(\Phi))$ *are satisfied. In this situation, for any* $p\leq n$, $u,u'\in I_p$, $\eta,\eta'\in\mathcal{P}(E_p)$ *and* $f\in Osc_1(E_n)$ *and any flow of masses and probability measures* $m=(m_n)_{n\geq0}\in\prod_{n\geq0}I_n$ *and* $\nu:=(\nu_n)_{n\geq0}\in\prod_{n\geq0}\mathcal{P}(E_n)$ *we have the following estimates*

$$\left|\Phi^1_{p,n,\nu'}(u')-\Phi^1_{p,n,\nu}(u)\right|\leq a^1_{p,n}\ |u-u'|+\sum_{p\leq q<n}\overline{a}^1_{q,n}\int\ \left|[\nu_q-\nu'_q](\varphi)\right|\ \Omega^1_{q+1,\nu'_q}(d\varphi)$$

$$\left|\Phi^2_{p,n,m'}(\eta')(f)-\Phi^2_{p,n,m}(\eta)(f)\right|\leq a^2_{p,n}\int\left|[\eta-\eta'](\varphi)\right|\Omega^2_{p,n,\eta'}(f,d\varphi)+\sum_{p\leq q<n}\overline{a}^2_{q,n}\ |m_q-m'_q|$$

with the collection of parameters $\overline{a}^i_{p,n}$, $i=1,2$, *defined in (3.6).*

Proof:
We use the decomposition

$$\begin{aligned}\Phi^1_{p,n,\nu'}(u')-\Phi^1_{p,n,\nu}(u)&=\Phi^1_{p,n,\nu}(u')-\Phi^1_{p,n,\nu}(u)\\&\quad+\sum_{p<q\leq n}\left[\Phi^1_{q,n,\nu}(\Phi^1_{p,q,\nu'}(u'))-\Phi^1_{q-1,n,\nu}(\Phi^1_{p,q-1,\nu'}(u'))\right]\end{aligned}$$

and the fact that

$$\begin{aligned}\Phi^1_{q-1,n,\nu}(\Phi^1_{p,q-1,\nu'}(u'))&=\Phi^1_{q,n,\nu}\left(\Phi^1_{q-1,q,\nu}\left[\Phi^1_{p,q-1,\nu'}(u')\right]\right)\\\Phi^1_{q,n,\nu}(\Phi^1_{p,q,\nu'}(u'))&=\Phi^1_{q,n,\nu}\left(\Phi^1_{q-1,q,\nu'}\left[\Phi^1_{p,q-1,\nu'}(u')\right]\right)\end{aligned}$$

and

$$\left|\Phi^1_{p,n,\nu}(u')-\Phi^1_{p,n,\nu}(u)\right|\leq a^1_{p,n}\ |u-u'|$$

and

$$\begin{aligned}&\left|\Phi^1_{q,n,\nu}(\Phi^1_{p,q,\nu'}(u'))-\Phi^1_{q-1,n,\nu}(\Phi^1_{p,q-1,\nu'}(u'))\right|\\&\leq a^1_{q,n}\ \left|\Phi^1_{q,\nu_{q-1}}\left[\Phi^1_{p,q-1,\nu'}(u')\right]-\Phi^1_{q,\nu'_{q-1}}\left[\Phi^1_{p,q-1,\nu'}(u')\right]\right|\\&\leq\ \overline{a}^1_{q-1,n}\ \int\ \left|[\nu_{q-1}-\nu'_{q-1}](\varphi)\right|\ \Omega_{q,\nu'_{q-1}}(d\varphi)\end{aligned}$$

to show that

$$\left|\Phi^1_{p,n,\nu'}(u') - \Phi^1_{p,n,\nu}(u)\right| \leq a^1_{p,n}\ |u-u'| + \sum_{p<q\leq n} \overline{a}^1_{q-1,n} \int\ \left|[\nu_{q-1} - \nu'_{q-1}](\varphi)\right|\ \Omega^1_{q,\nu'_{q-1}}(d\varphi)$$

In the same way, we use the decomposition

$$\begin{aligned}\left[\Phi^2_{p,n,m'}(\eta') - \Phi^2_{p,n,m}(\eta)\right] &= \left[\Phi^2_{p,n,m}(\eta') - \Phi^2_{p,n,m}(\eta)\right] \\ &\quad + \sum_{p<q\leq n} \left[\Phi^2_{q,n,m}(\Phi^2_{p,q,m'}(\eta')) - \Phi^2_{q-1,n,m}(\Phi^2_{p,q-1,m'}(\eta'))\right]\end{aligned}$$

and the fact that

$$\begin{aligned}\Phi^2_{q-1,n,m}(\Phi^2_{p,q-1,m'}(\eta')) &= \Phi^2_{q,n,m}\left(\Phi^2_{q-1,q,m}\left[\Phi^2_{p,q-1,m'}(\eta')\right]\right) \\ \Phi^2_{q,n,m}(\Phi^2_{p,q,m'}(\eta')) &= \Phi^2_{q,n,m}\left(\Phi^2_{q-1,q,m'}\left[\Phi^2_{p,q-1,m'}(\eta')\right]\right)\end{aligned}$$

and

$$\left|\Phi^2_{p,n,m}(\eta')(f) - \Phi^2_{p,n,m}(\eta)(f)\right| \leq a^2_{p,n} \int\ \left|[\eta-\eta'](\varphi)\right|\ \Omega^2_{p,n,\eta'}(f,d\varphi)$$

to show that

$$\begin{aligned}&\left|\Phi^2_{q,n,m}(\Phi^2_{p,q,m'}(\eta')) - \Phi^2_{q-1,n,m}(\Phi^2_{p,q-1,m'}(\eta'))\right| \\ &\leq a^2_{q,n} \int\ \left|[\Phi^2_{q,m_{q-1}}\left[\Phi^2_{p,q-1,m'}(\eta')\right] - \Phi^2_{q,m'_{q-1}}\left[\Phi^2_{p,q-1,m'}(\eta')\right]](\varphi)\right|\ \Omega^2_{q,n,\Phi^2_{p,q,m'}(\eta')}(f,d\varphi) \\ &\leq \overline{a}^2_{q-1,n}\ |m_{q-1} - m'_{q-1}|\end{aligned}$$

Using these estimates we conclude that

$$\left|\left[\Phi^2_{p,n,m'}(\eta') - \Phi^2_{p,n,m}(\eta)\right](f)\right| \leq a^2_{p,n} \int\ \left|[\eta-\eta'](\varphi)\right|\ \Omega^2_{p,n,\eta}(f,d\varphi) + \sum_{p<q\leq n} \overline{a}^2_{q-1,n}\ |m_{q-1} - m'_{q-1}|$$

This ends the proof of the lemma. ■

Now we come to the proof of proposition 3.3.

Proof of proposition 3.3:

We fix a parameter $p \geq 0$, and we let $(m_n)_{n\geq p}, (m'_n)_{n\geq p} \in \prod_{n\geq p} I_n$ and $(\nu_n)_{n\geq p}$, and $(\nu'_n)_{n\geq p} \in \prod_{n\geq p} \mathcal{P}(E_n)$ be defined by the following recursive formulae

$$\forall q > p \qquad m'_q = \Phi^1_{q,\nu'_{q-1}}(m'_{q-1}) \quad \text{and} \quad \nu'_q = \Phi^2_{q,m'_{q-1}}(\nu'_{q-1})$$

$$\forall q > p \qquad m_q = \Phi^1_{q,\nu_{q-1}}(m_{q-1}) \quad \text{and} \quad \nu_q = \Phi^2_{q,m_{q-1}}(\nu_{q-1})$$

with the initial condition for $q = p$

$$(\nu_p, \nu'_p) = (\eta, \eta') \quad \text{and} \quad (m_p, m'_p) = (u, u')$$

By construction, we have

$$\nu'_q = \Phi^2_{p,q,m'}(\eta') \quad \text{and} \quad \nu_q = \Phi^2_{p,q,m}(\eta)$$

as well as

$$m'_q = \Phi^1_{p,q,\nu'}(u') \quad \text{and} \quad m_q = \Phi^1_{p,q,\nu}(u)$$

In this case, using lemma 5.1 it follows that

$$\begin{aligned}
&\left|\left[\Gamma^2_{p,n}(m',\eta') - \Gamma^2_{p,n}(m,\eta)\right](f)\right| \\
&\leq a^2_{p,n} \int \ \left|[\eta-\eta'](\varphi)\right| \ \Omega^2_{p,n,\eta'}(f,d\varphi) + \sum_{p\leq q<n} \ \overline{a}^2_{q,n} \ |\Gamma^1_{p,q}(m',\eta') - \Gamma^1_{p,q}(m,\eta)|
\end{aligned}$$

and

$$\begin{aligned}
&\left|\Gamma^1_{p,n}(m',\eta') - \Gamma^1_{p,n}(m,\eta)\right| \\
&\leq a^1_{p,n} \ |m-m'| + \sum_{p\leq q<n} \ \overline{a}^1_{q,n} \ \int \ \left|[\Gamma^2_{p,q}(m',\eta') - \Gamma^2_{p,q}(m,\eta)](\varphi)\right| \ \overline{\Omega}^1_{p,q,m',\eta'}(d\varphi)
\end{aligned}$$

with the probability measure $\overline{\Omega}^1_{p,q,m',\eta'} = \Omega^1_{q+1,\Gamma^2_{p,q}(m',\eta')}$.

Combining these two estimates, we arrive at the following inequality

$$\begin{aligned}
&\left|\left[\Gamma^2_{p,n}(m',\eta') - \Gamma^2_{p,n}(m,\eta)\right](f)\right| \\
&\leq a^2_{p,n} \int \ \left|[\eta-\eta'](\varphi)\right| \ \Omega^2_{p,n,\eta'}(f,d\varphi) + \left[\sum_{p\leq q<n} \ a^1_{p,q} \ \overline{a}^2_{q,n}\right] \ |m-m'| \\
&+ \sum_{p\leq r<q<n} \ \overline{a}^1_{r,q} \ \overline{a}^2_{q,n} \ \int \ \left|[\Gamma^2_{p,r}(m',\eta') - \Gamma^2_{p,r}(m,\eta)](\varphi)\right| \ \overline{\Omega}^1_{p,r,m',\eta'}(d\varphi)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left|\left[\Gamma^2_{p,n}(m',\eta') - \Gamma^2_{p,n}(m,\eta)\right](f)\right| \\
&\leq b'_{p,n} \ |m-m'| + a^2_{p,n} \int \ \left|[\eta-\eta'](\varphi)\right| \ \Omega^2_{p,n,\eta'}(f,d\varphi) \\
&+ \sum_{p\leq r_1<n} \ b_{r_1,n} \ \int \ \left|[\Gamma^2_{p,r_1}(m',\eta') - \Gamma^2_{p,r_1}(m,\eta)](\varphi)\right| \ \overline{\Omega}^1_{p,r_1,m',\eta'}(d\varphi)
\end{aligned}$$

Our next objective is to show that

$$\begin{aligned}
&\left|\left[\Gamma^2_{p,n}(m',\eta') - \Gamma^2_{p,n}(m,\eta)\right](f)\right| \\
&\leq \alpha^k_{p,n} \ |m-m'| + \beta^k_{p,n} \int \ \left|[\eta-\eta'](\varphi)\right| \ \Theta^k_{p,n,\eta'}(f,d\varphi) \\
&+ \sum_{p\leq r_1<r_2<\ldots<r_k<n} \ b_{r_1,r_2} \ \ldots \ b_{r_k,n} \ \int \ \left|[\Gamma^2_{p,r_1}(m',\eta') - \Gamma^2_{p,r_1}(m,\eta)](\varphi)\right| \ \overline{\Omega}^1_{p,r_1,m',\eta'}(d\varphi)
\end{aligned}$$

for any $k \leq (n-p)$ for some Markov transitions $\Theta^k_{p,n,m'\eta'}(f,d\varphi)$ and the parameters

$$\begin{aligned}
\alpha^k_{p,n} &= b'_{p,n} + \sum_{l=1}^{k-1} \sum_{p\leq r_1<\ldots r_l<n} b'_{p,r_1}\ b_{r_1,r_2}\ldots b_{r_l,n} \\
\beta^k_{p,n} &= a^2_{p,n} + \sum_{l=1}^{k-1} \sum_{p\leq r_1<\ldots r_l<n} a^2_{p,r_1}\ b_{r_1,r_2}\ldots b_{r_l,n}
\end{aligned}$$

We proceed by induction on the parameter k. Firstly, we observe that the result is satisfied for $k=1$ with

$$\left(\alpha^1_{p,n},\beta^1_{p,n}\right) = \left(b'_{p,n},a^2_{p,n}\right) \quad \text{and} \quad \Theta^1_{p,n,\eta'} = \Omega^2_{p,n,\eta'}$$

We further assume that the result is satisfied at rank k. In this situation, using the fact that

$$\begin{aligned}
&\left|\left[\Gamma^2_{p,r_1}(m',\eta') - \Gamma^2_{p,r_1}(m,\eta)\right](\varphi)\right] \\
&\leq b'_{p,r_1}\ |m-m'| + a^2_{p,r_1} \int\ \left|[\eta-\eta'](\varphi')\right|\ \Omega^2_{p,r_1,\eta'}(\varphi,d\varphi') \\
&+ \sum_{p\leq r_0<r_1} b_{r_0,r_1}\ \int\ \left|[\Gamma^2_{p,r_0}(m',\eta') - \Gamma^2_{p,r_0}(m,\eta)](\varphi)\right|\ \overline{\Omega}^1_{p,r_0,m',\eta'}(d\varphi)
\end{aligned}$$

we conclude that

$$\begin{aligned}
&\left|\left[\Gamma^2_{p,n}(m',\eta') - \Gamma^2_{p,n}(m,\eta)\right](f)\right] \\
&\leq \alpha^{k+1}_{p,n}\ |m-m'| + \beta^{k+1}_{p,n} \int\ \left|[\eta-\eta'](\varphi)\right|\ \Theta^{k+1}_{p,n,m'\eta'}(f,d\varphi) \\
&+ \sum_{p\leq r_0<r_1<r_2<\ldots<r_k<n} b_{r_0,r_1}\ b_{r_1,r_2}\ \ldots\ b_{r_k,n} \int\ \left|[\Gamma^2_{p,r_0}(m',\eta') - \Gamma^2_{p,r_0}(m,\eta)](\varphi)\right|\ \overline{\Omega}^1_{p,r_0,m',\eta'}(d\varphi)
\end{aligned}$$

with

$$\begin{aligned}
\alpha^{k+1}_{p,n} &= \alpha^k_{p,n} + \sum_{p\leq r_1<r_2<\ldots<r_k<n} b'_{p,r_1}\ b_{r_1,r_2}\ \ldots\ b_{r_k,n} \\
\beta^{k+1}_{p,n} &= \beta^k_{p,n} + \sum_{p\leq r_1<r_2<\ldots<r_k<n} a^2_{p,r_1}\ b_{r_1,r_2}\ \ldots\ b_{r_k,n}
\end{aligned}$$

and the Markov transition

$$\begin{aligned}
\beta^{k+1}_{p,n}\ \Theta^{k+1}_{p,n,m'\eta'}(f,d\varphi) &= \beta^k_{p,n}\ \Theta^k_{p,n,\eta'}(f,d\varphi) \\
&+ \sum_{p\leq r_1<r_2<\ldots<r_k<n} a^2_{p,r_1}\ b_{r_1,r_2}\ \ldots\ b_{r_k,n}\ \left(\overline{\Omega}^1_{p,r_1,m',\eta'}\Omega^2_{p,r_1,\eta'}\right)(d\varphi)
\end{aligned}$$

We end the proof of the proposition using the fact that

$$\begin{aligned}
\left|\Gamma^1_{p,n}(m',\eta') - \Gamma^1_{p,n}(m,\eta)\right| \ \leq\ & \left[a^1_{p,n} + \sum_{p\leq q<n} c^{2,1}_{p,q}\ \overline{a}^1_{q,n}\right]\ |m-m'| \\
&+ \sum_{p\leq q<n} \overline{a}^1_{q,n}\ c^{2,2}_{p,q} \int\ \left|[\eta-\eta'](\varphi')\right|\ \left[\overline{\Omega}^1_{p,q,m',\eta'}\Theta_{p,q,\eta'}\right](d\varphi')
\end{aligned}$$

This proof of the proposition is now completed. ■

5.4 Proof of theorem 3.11

For any $\eta \in \mathcal{P}(E)$ and any $u, u' \in I_n$, we have

$$\begin{aligned}
&\left|\Phi^1_{n+1,\eta}(u) - \Phi^1_{n+1,\eta}(u')\right| \\
&= |u-u'| \left[r(1-d) + rdh \int \mathcal{Y}_n(dy) \; \frac{\eta(g(\centerdot,y))}{[h+du\eta(g(\centerdot,y))][h+du'\eta(g(\centerdot,y))]}\right] \\
&\leq |u-u'| \left[r(1-d) + rdh \; \mathcal{Y}_n\left(\frac{g^+}{[h+dm^-g^-)]^2}\right)\right]
\end{aligned}$$

This implies that condition (3.2) is satisfied with

$$a^1_{n,n+1} \leq r(1-d) + rdh \; \mathcal{Y}_n\left(\frac{g^+}{[h+dm^-g^-)]^2}\right)$$

In the same way, for any $\eta, \eta' \in \mathcal{P}(E)$ and any $u \in I_n$, we have

$$\Phi^1_{n+1,\eta}(u) - \Phi^1_{n+1,\eta'}(u) = rdhu \int \mathcal{Y}_n(dy) \; \frac{1}{[h+du\eta(g(\centerdot,y))][h+du\eta'(g(\centerdot,y))]} \; (\eta - \eta')\,(g(.,y))$$

$$\tau^1_{n+1} \leq rdhm^+ \; \mathcal{Y}_n\left(\frac{g^+ - g^-}{[h+dm^-g^-]^2}\right)$$

and the probability measure

$$\Omega^1_{n,\eta'}(d\varphi) \propto \int \mathcal{Y}_n(dy) \; \frac{g^+(y) - g^-(y)}{[h+dm^-g^-(y)]^2} \; \delta_{\frac{g(\centerdot,y)}{g^+(y)-g^-(y)}}(d\varphi)$$

Now, we come to the analysis of the mappings

$$\Phi^2_{n+1,u}(\eta) \propto r(1-d)u \; \eta M + \int \mathcal{Y}_n(dy) \; w_u(\eta,y) \; \Psi_{g(\centerdot,y)}(\eta) M + \mu(1) \; \overline{\mu}$$

with the weight functions

$$w_u(\eta,y) := \frac{rdu\eta(g(.,y))}{h+du\eta(g(.,y))} = r\left(1 - \frac{h}{h+du\eta(g(.,y))}\right)$$

Notice that

$$w^-(y) := \frac{rdm^-g^-(y)}{h+dm^-g^-(y)} \leq w_u(\eta,y) \leq w^+(y) := \frac{rdm^+g^+(y)}{h+dm^+g^+(y)}$$

To have a more synthetic formula, we extend the observation state space with two auxiliary points c_1, c_2 and we set

$$\mathcal{Y}^c_n = \mathcal{Y}_n + \delta_{c_1} + \delta_{c_2}$$

we extend the likelihood and the weight functions by setting

$$g(x,c_1) = g(x,c_2) = 1$$

and

$$\begin{aligned} w^-(c_1) &:= r(1-d)m^- \leq w_u(\eta,c_1) := r(1-d)u \leq w^+(c_1) := r(1-d)m^+ \\ w_u(\eta,c_2) &= w^+(c_2) = w^-(c_2) := \mu(1) \end{aligned}$$

In this notation, we find that

$$\Phi^2_{n+1,u}(\eta) \propto \int \mathcal{Y}^c_n(dy)\ w_u(\eta,y)\ \Psi_{g(\,.\,,y)}(\eta) M_y$$

with the collection of Markov transitions M_y defined below

$$\forall y \not\in \{c_2\} \qquad M_y = M \quad \text{and} \quad M_{c_2} = \overline{\mu}$$

Notice that the normalizing constants $\mathcal{Y}^c_n(w_u(\eta,.))$ satisfy the following lower bounds

$$\mathcal{Y}^c_n(w_u(\eta,.)) \ \ \geq\ \ \mathcal{Y}^c_n(w^-) = r(1-d)\ m^- + \mathcal{Y}_n\left(w^-\right) + \mu(1)$$

We analyze the Lipschitz properties of the mappings $\Phi^2_{n+1,u}$ using the following decomposition

$$\Phi^2_{n+1,u}(\eta) - \Phi^2_{n+1,u}(\eta') = \Delta_{n+1,u}(\eta,\eta') + \Delta'_{n+1,u}(\eta,\eta')$$

with the signed measures

$$\Delta_{n+1,u}(\eta,\eta') = \int \mathcal{Y}^c_n(dy)\ \frac{w_u(\eta,y)}{\mathcal{Y}^c_n(w_u(\eta,.))}\ \left[\Psi_{g(\,.\,,y)}(\eta)M_y - \Psi_{g(\,.\,,y)}(\eta')M_y\right]$$

and

$$\Delta'_{n+1,u}(\eta,\eta') = \frac{1}{\mathcal{Y}^c_n(w_u(\eta,.))} \int \mathcal{Y}^c_n(dy)\ \left[w_u(\eta,y) - w_u(\eta',y)\right]\ \left(\Psi_{g(\,.\,,y)}(\eta')M_y - \Phi^2_{n+1,u}(\eta')\right)$$

Arguing as in the proof of theorem 3.7 given in the appendix (see for instance (5.2)), one checks that

$$\begin{aligned} &|\Delta_{n+1,u}(\eta,\eta')(f)| \\ &\leq \tfrac{1}{\mathcal{Y}^c_n(w^-)}\ \left(r(1-d)m^+\ |(\eta-\eta')(M(f))| + \textstyle\int \mathcal{Y}_n(dy)\ w^+(y)\ \frac{g^+(y)}{g^-(y)}\ \left|(\eta-\eta')(S^y_{\eta'}M(f))\right|\right) \end{aligned}$$

for some collection of Markov transitions $S^y_{\eta'}$ from E into itself. It is also readily checked that

$$\left|\Delta'_{n+1,u}(\eta,\eta')(f)\right| \leq \frac{hrdm^+}{\mathcal{Y}^c_n(w^-)}\ \int \mathcal{Y}_n(dy) \frac{1}{(h+m^-dg^-(y))^2}\ |(\eta-\eta')(g(.,y))|$$

This clearly implies that condition (3.3) is satisfied with

$$a^2_{n,n+1} \leq \frac{1}{\mathcal{Y}^c_n(w^-)}\ \left(\beta(M)\left[r(1-d)m^+ + \mathcal{Y}_n\left(\frac{w^+g^+}{g^-}\right)\right] + hrdm^+\mathcal{Y}_n\left(\frac{g^+-g^-}{(h+m^-dg^-)^2}\right)\right)$$

We analyze the continuity properties of the mappings $u \mapsto \Phi^2_{n+1,u}(\eta)$ using the following decomposition

$$
\begin{aligned}
&\Phi^2_{n+1,u}(\eta) - \Phi^2_{n+1,u'}(\eta)\\
&= \tfrac{1}{\mathcal{Y}_n^c(w_u(\eta,\cdot))} \int \mathcal{Y}_n^c(dy)\ \ [w_u(\eta,y) - w_{u'}(\eta,y)]\ \ \left(\Psi_{g(\cdot,y)}(\eta)M_y - \Phi^2_{n+1,u'}(\eta)\right)
\end{aligned}
$$

This implies that

$$
\left|\left[\Phi^2_{n+1,u}(\eta) - \Phi^2_{n+1,u'}(\eta)\right](f)\right| \leq \tfrac{1}{\mathcal{Y}_n^c(w^-)} \left[r(1-d) + hrd\ \mathcal{Y}_n\left(\tfrac{g^+}{(h+dm^-g^-)^2}\right)\right]\ \ |u-u'|
$$

This shows that condition (3.5) is satisfied with

$$
\tau^2_{n+1} \leq \frac{1}{\mathcal{Y}_n^c(w^-)} \left[r(1-d) + hrd\ \mathcal{Y}_n\left(\frac{g^+}{(h+dm^-g^-)^2}\right)\right]
$$

This ends the proof of the theorem. ■

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