CPHD filtering in unknown clutter rate and detection profile

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***Abstract*—In Bayesian multi-target filtering we have to contend with two notable sources of uncertainty, clutter and detection. Knowledge of parameters such as clutter rate and detection profile are of critical importance in multi-target filters such as the probability hypothesis density (PHD) and Cardinalized PHD (CPHD) filters. Naive application of the CPHD (and PHD) filter with mismatches in clutter and detection model parameters results in biased estimates. In this paper we show how to use the CPHD (and PHD) filter in unknown clutter rate and detection profile.**

***Index Terms*—PHD, CPHD, multi-target tracking, parameter estimation, robust filtering, finite set statistics.**

I. Introduction

To alleviate the intractability of the multi-target Bayes filter [1], the probability hypothesis density (PHD) and subsequently Cardinalized PHD (CPHD) filters have been proposed [2], [3]. These filters operate on the single-target state space and avoid the combinatorial problem that arises from data association. Since their inceptions the PHD and CPHD filters have generated substantial interest from academia as well as the commercial sector with the developments of numerical solutions such as sequential Monte Carlo (SMC) or particle and Gaussian mixtures [4]–[6]. Extension to linear Jump Markov multi-target models for tracking maneuvering targets has been proposed in [7]. Recently, important developments such as the auxiliary particle-PHD filter [8], and measurement-oriented particle labeling technique [9], partially solve the clustering problem in the extraction of state estimates from the particle population. Clever uses of the PHD filter with measurement-driven birth intensity were independently proposed in [9] and [10] to improve tracking performance as well as obviating exact knowledge of the birth intensity.

In PHD/CPHD filtering, no less than multi-target filtering in general, we have to contend with two notable sources of uncertainty, clutter and detection, in addition to the process and measurement noise from each target. Clutter are spurious measurements that do not belong to any target, while detection uncertainty refers to the phenomena that the sensor does not always detect the targets. Knowledge of parameters such as clutter rate and detection profile are of critical importance in Bayesian multi-target filtering, arguably, more so than measurement noise model in single-target filtering. Unfortunately, except for a few application areas, exact knowledge of these model parameters is not available. For example, in visual tracking measurements are extracted from images via various background or foreground modeling techniques [11] and the missed detection and false detection processes vary with the detection method. A major problem is the time-varying nature of these processes. Consequently, there is no guarantee that the parameters chosen from training data will be sufficient for the multi-object filter at subsequent frames.Thus the ability of the PHD and CPHD filters to accommodate mismatch in clutter rate and detection profile is very important in practice.

In this paper we show how to use the CPHD (and PHD) filter correctly when the clutter rate and/or detection profile are not known. In particular we show that the CPHD (and PHD) recursion can adaptively learn non-uniform detection profile or/and clutter rate while filtering, provided that the detection profile and clutter background do not change too rapidly compared to the measurement-update rate. Analytic approximate solutions to these filters for linear Gaussian multi-target models are proposed in a more comprehensive study [12]. These filters are implementable special cases of the general theoretical approach described in references [13], [14].

A related work is given in [15] which deals with the problem of calibrating time-invariant multi-target model parameters. The key idea is to find the vector of parameters that maximizes an approximate marginal likelihood of the observed data via gradient ascent. Neither the approximate likelihood function nor its gradient are computable and the authors have proposed an SMC approximation. An even more closely related work which investigates clutter estimation while filtering (with known detection profile) using the PHD filter is given in [16].

II. The CPHD Recursion

Suppose that at time k, there are $N(k)$ target states $x_{k,1},\ldots,x_{k,N(k)}$, each taking values in a state space $\mathcal{X}$, and $M(k)$ observations $z_{k,1},\ldots,z_{k,M(k)}$ each taking values in an observation space $\mathcal{Z}$. Then, the multi-target state and multi-target observations, at time k, are the finite sets [1], [2]

$$X_k = \{x_{k,1},\ldots,x_{k,N(k)}\} \subset \mathcal{X},$$
$$Z_k = \{z_{k,1},\ldots,z_{k,M(k)}\} \subset \mathcal{Z},$$

The following notation is used throughout. Denote by C_j^{ℓ} the *binomial coefficient* $\frac{\ell!}{j!(\ell-j)!}$, P_j^n the *permutation coefficient*

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$\frac{n!}{(n-j)!}$, $\langle\cdot,\cdot\rangle$ the *inner product* defined between two real valued functions α and β by $\langle\alpha,\beta\rangle = \int \alpha(x)\beta(x)dx$,(or $\sum_{\ell=0}^{\infty}\alpha(\ell)\beta(\ell)$ when α and β are real sequences), and $e_j(\cdot)$ the *elementary symmetric function* of order j defined for a finite set Z of real numbers by $e_j(Z) = \sum_{S\subseteq Z,|S|=j}\left(\prod_{\zeta\in S}\zeta\right)$,with $e_0(Z) = 1$ by convention and $|S|$ the cardinality of a set S.

The CPHD recursion rests on the following assumptions regarding the target dynamics and observations:

- Each target evolves and generates measurements independently of one another;
- The birth RFS and the surviving RFSs are independent of each other;
- The clutter RFS is an i.i.d cluster process and independent of the measurement RFSs;
- The prior and predicted multi-target RFSs are i.i.d cluster processes.

Let $v_{k|k-1}$ and $\rho_{k|k-1}$ denote the intensity and cardinality distribution associated with the predicted multi-target state, and let v_k and ρ_k denote the intensity and cardinality distribution associated with the posterior multi-target state. The intensity and cardinality distribution are summary statistics of the underlying RFS. The intensity of an RFS is analogous to the mean of a random variable while the cardinality distribution describes, probabilistically, the number of elements in an RFS. The following propositions, which constitute the prediction and update step of the CPHD filter, show how v_k and ρ_k are jointly propagated in time [3], [6].

Proposition 1 *If at time $k-1$, the posterior cardinality distribution ρ_{k-1} and posterior intensity v_{k-1} are given, then the predicted cardinality distribution $\rho_{k|k-1}$ and predicted intensity $v_{k|k-1}$ are given by*

$$\rho_{k|k-1}(n) = \sum_{j=0}^{n}\rho_{\Gamma,k}(n-j)\Pi_{k|k-1}[v_{k-1},\rho_{k-1}](j), \quad (1)$$

$$v_{k|k-1}(x) = \gamma_k(x) + \int p_{S,k}(\zeta)f_{k|k-1}(x|\zeta)v_{k-1}(\zeta)d\zeta, \quad (2)$$

where

$$\Pi_{k|k-1}[v,\rho](j) = \sum_{\ell=j}^{\infty}C_j^{\ell}\rho(\ell)\frac{\langle p_{S,k},v\rangle^{j}\langle 1-p_{S,k},v\rangle^{\ell-j}}{\langle 1,v\rangle^{\ell}},$$

$$\begin{aligned}
\rho_{\Gamma,k}(\cdot) &= \textit{cardinality distribution of birth RFS},\\
\gamma_k(\cdot) &= \textit{intensity function of birth RFS},\\
p_{S,k}(\zeta) &= \textit{probability of survival to time } k \textit{ given}\\
&\quad\ \textit{state } \zeta \textit{ at time } k-1,\\
f_{k|k-1}(x|\zeta) &= \textit{single target Markov transition density}\\
&\quad\ \textit{from } k-1 \textit{ to time } k.
\end{aligned}$$

Proposition 2 *If at time k, the predicted cardinality distribution $\rho_{k|k-1}$ and predicted intensity $v_{k|k-1}$ are given, then for a given measurement set Z_k, the updated cardinality distribution ρ_k and updated intensity v_k are given by*

$$\rho_k(n) = \frac{\Upsilon_k^0[v_{k|k-1};Z_k](n)\rho_{k|k-1}(n)}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle}, \quad (3)$$

$$\begin{aligned}
v_k(x) = \Bigg[&q_{D,k}(x)\frac{\langle\Upsilon_k^1[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle}\\
&+\sum_{z\in Z_k}\psi_{k,z}(x)\frac{\langle\Upsilon_k^1[v_{k|k-1};Z_k-\{z\}],\rho_{k|k-1}\rangle}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],\rho_{k|k-1}\rangle}\Bigg]v_{k|k-1}(x),
\end{aligned} \quad (4)$$

where

$$\Upsilon_k^u[v,Z](n) = \sum_{j=0}^{\min(|Z|,n)}(|Z|-j)!\rho_{K,k}(|Z|-j)P_{j+u}^{n}\times \frac{\langle 1-p_{D,k},v\rangle^{n-(j+u)}}{\langle 1,v\rangle^{n}}e_j(\Xi_k(v,Z)) \quad (5)$$

$$\psi_{k,z}(x) = \frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)}g_k(z|x)p_{D,k}(x), \quad (6)$$

$$\Xi_k(v,Z) = \left\{\langle v,\psi_{k,z}\rangle : z\in Z\right\}, \quad (7)$$

$$\begin{aligned}
p_{D,k}(x) &= \textit{probability of detection of state } x \textit{ at time } k,\\
q_{D,k}(x) &= 1-p_{D,k}(x),\\
g_k(z|x) &= \textit{single target measurement likelihood at time } k,\\
\rho_{K,k}(\cdot) &= \textit{cardinality distribution of clutter at time } k,\\
\kappa_k(\cdot) &= \textit{intensity function of clutter RFS at time } k,
\end{aligned}$$

If the cardinalities of the RFS involved are Poisson, then the above propositions reduce to the PHD recursion

$$v_{k|k-1}(x) = \gamma_k(x) + \int p_{S,k}(\zeta)f_{k|k-1}(x|\zeta)v_{k-1}(\zeta)d\zeta,$$

$$v_k(x) = \left[q_{D,k}(x) + \sum_{z\in Z_k}\frac{p_{D,k}(x)g_k(z|x)}{\kappa_k(z)+\langle p_{D,k}g_k(z|\cdot),v_{k|k-1}\rangle}\right]\times v_{k|k-1}(x).$$

III. CPHD Filtering with Unknown Clutter Rate

The underlying idea in this development is to model clutter by a random finite set of ‘false targets’ or ‘generator objects’ (distinct from actual targets) which are specified by standard type models for births/deaths and transitions as well as misses/detections and measurements (also distinct from the respective models for actual targets). The multi target state is then the finite set of actual targets and clutter generators, which is to be estimated from the sequence of finite sets of observations. Estimation of the hybrid state of targets/objects then yields information on the number and individual states of actual targets in addition to the unknown clutter rate. In the following, we expand the CPHD recursion, which propagates separate intensity functions for the two target types, jointly alongside the cardinality distribution of all targets/objects, i.e. both actual and clutter.

A. Hybrid State Space Model

Let $\mathcal{X}^{(1)}$ denote the state space for actual targets, $\mathcal{X}^{(0)}$ denote the state space for clutter generators, and $\mathcal{Z}$ denote a common observation space. Define the hybrid state space

$$\ddot{\mathcal{X}} = \mathcal{X}^{(1)} \uplus \mathcal{X}^{(0)},$$

where $\uplus$ denotes a disjoint union. The double dot notation is used throughout to denote a function or variable defined on the hybrid state space, i.e. denote $\ddot{x} \in \ddot{\mathcal{X}}$ for a hybrid state as opposed to $x \in \mathcal{X}^{(1)}$ or $c \in \mathcal{X}^{(0)}$ for actual or clutter states. The integral of a function $\ddot{f} : \ddot{\mathcal{X}} \to \mathbb{R}$ is given by

$$\int_{\ddot{\mathcal{X}}} \ddot{f}(\ddot{x}) d\ddot{x} = \int_{\mathcal{X}^{(1)}} \ddot{f}(x) dx + \int_{\mathcal{X}^{(0)}} \ddot{f}(c) dc,$$

It is assumed throughout that actual targets and clutter objects are statistically independent. Where necessary, a superscript $^{(1)}$ is used to denote functions or variables pertaining to actual targets, while a superscript $^{(0)}$ is used to denote functions or variables on the space of clutter objects. For any space $\mathcal{X}$, let $\mathcal{F}(\mathcal{X})$ denotes the set of all finite subsets of $\mathcal{X}$.

Suppose at time $k-1$ that hybrid multi-target state $\ddot{X}_{k-1} \in \mathcal{F}(\ddot{\mathcal{X}})$ is given by the disjoint union of actual and clutter states respectively, i.e. $\ddot{X}_{k-1} = X_{k-1}^{(1)} \uplus X_{k-1}^{(0)}$ where $X_{k-1}^{(1)} \in \mathcal{F}(\mathcal{X}^{(1)})$ and $X_{k-1}^{(0)} \in \mathcal{F}(\mathcal{X}^{(0)})$. At time k, the multi-target state evolves to $\ddot{X}_k \in \mathcal{F}(\ddot{\mathcal{X}})$ and is given by the disjoint union of finite set states of transitioned actual targets and finite set states of clutter generators respectively at time k, i.e.

$$\ddot{X}_k = X_k^{(1)} \uplus X_k^{(0)}. \tag{8}$$

where $X_k^{(1)} \in \mathcal{F}(\mathcal{X}^{(1)})$ and $X_k^{(0)} \in \mathcal{F}(\mathcal{X}^{(0)})$. The actual multi-target state and clutter multi-target state at time k are given by the union of surviving states and new births respectively, i.e.

$$X_k^{(1)} = \bigcup_{x_{k-1} \in X_{k-1}^{(1)}} S_{k|k-1}^{(1)}(x_{k-1}) \cup \Gamma_k^{(1)}, \tag{9}$$

$$X_k^{(0)} = \bigcup_{c_{k-1} \in X_{k-1}^{(0)}} S_{k|k-1}^{(0)}(c_{k-1}) \cup \Gamma_k^{(0)}, \tag{10}$$

and $S_{k|k-1}^{(1)}(x_{k-1})$ is an RFS which takes on the empty set $\emptyset$ with probability $1 - p_{S,k}^{(1)}(x_{k-1})$ or a singleton $\{x_k\} \in \mathcal{X}^{(1)}$ with probability density $p_{S,k}^{(1)}(x_{k-1}) f_{k|k-1}^{(1)}(x_k|x_{k-1})$. Similarly, $S_{k|k-1}^{(0)}(c_{k-1})$ is an RFS which takes on the empty set $\emptyset$ with probability $1 - p_{S,k}^{(0)}(c_{k-1})$ or a singleton $\{c_k\} \in \mathcal{X}^{(0)}$ with probability density $p_{S,k}^{(0)}(c_{k-1}) f_{k|k-1}^{(0)}(c_k|c_{k-1})$. Also, $\Gamma_k^{(1)}$ and $\Gamma_k^{(0)}$ are RFSs with realizations in $\mathcal{X}^{(1)}$ and $\mathcal{X}^{(0)}$ respectively of new births for actual and clutter targets, described by intensity functions $\gamma_k^{(1)}$ and $\gamma_k^{(0)}$ and cardinality distributions $\rho_{\Gamma,k}^{(1)}$ and $\rho_{\Gamma,k}^{(0)}$ respectively. Consequently, the RFS of all (actual and clutter) target births $\Gamma_k^{(1)} \uplus \Gamma_k^{(0)}$ has cardinality distribution $\ddot{\rho}_{\Gamma,k} = \rho_{\Gamma,k}^{(1)} * \rho_{\Gamma,k}^{(0)}$ (where $*$ denotes a convolution) and intensity function $\ddot{\gamma}_k = \gamma_k^{(1)} + \gamma_k^{(0)}$. It is assumed that conditional on $X_{k-1}^{(1)}$ and $X_{k-1}^{(0)}$ respectively, the RFSs $S_{k|k-1}^{(1)}(\cdot)$ and $S_{k|k-1}^{(0)}(\cdot)$ are statistically independent. Note that actual targets cannot become clutter objects and vice-versa. *To simplify notations, it is assumed that clutter generators are identical, and hence it is possible to ignore any functional dependence on the actual state of a clutter generator. A theoretically more general approach that does not rely on these assumptions, is presented in [13], [14].*

At time k, the hybrid multi-target state $\ddot{X}_k$ produces a finite set of measurements $Z_k \in \mathcal{F}(\mathcal{Z})$ given by the union of measurements produced by actual and clutter states, i.e.

$$Z_k = D_k^{(1)}(X_k^{(1)}) \cup D_k^{(0)}(X_k^{(0)}), \tag{11}$$

where

$$D_k^{(1)}(X_k^{(1)}) = \bigcup_{x_k \in X_k^{(1)}} \Theta_k^{(1)}(x_k), \tag{12}$$

$$D_k^{(0)}(X_k^{(0)}) = \bigcup_{i=1,\dots,|X_k^{(0)}|} \Theta_{k,i}^{(0)}, \tag{13}$$

with $\Theta_k^{(1)}(x_k)$ being an RFS which takes on the empty set $\emptyset$ with probability $q_{D,k}^{(1)}(x_k) = 1 - p_{D,k}^{(1)}(x_k)$ or a singleton $\{z_k\} \in \mathcal{Z}$ with probability density $p_{D,k}^{(1)}(x_k) g_k(z_k|x_k)$, and for each $i = 1, \dots, |X_k^{(0)}|$, $\Theta_{k,i}^{(0)}$ is an RFS which takes on the empty set $\emptyset$ with probability $q_{D,k}^{(0)} = 1 - p_{D,k}^{(0)}$ or a singleton $\{z_k\} \in \mathcal{Z}$ with probability density $p_{D,k}^{(0)} \varkappa_k(z_k)$. It is implicitly assumed that conditional on X_k, the RFSs in the union of (12) and (13) are statistically independent. *Since clutter generators are identical, they have the same spatial measurement distribution, can generate at most one return at each time, and have the same detection or generation probability. These are reasonable modelling assumptions considering the little amount of available statistical information on clutter. Note the difference between the clutter model adopted here and the standard one: clutter is not Poisson but is binomial (given by clutter generators).*

B. Recursion

The CPHD filter with unknown clutter rate jointly propagates the posterior intensity $\ddot{v}_k(\cdot)$ and posterior cardinality distribution $\ddot{\rho}_k(\cdot)$ of the hybrid state $\ddot{X}_k$. Due to the construction of the hybrid state space as a disjoint union of actual target and clutter generator spaces, the intensity $\ddot{v}_k(\cdot)$ is decomposable into

$$\ddot{v}_k(\ddot{x}) = \begin{cases} v_k^{(1)}(x), & \ddot{x} = x \\ v_k^{(0)}(c), & \ddot{x} = c \end{cases}$$

where $v_k^{(1)}(\cdot)$ and $v_k^{(0)}(\cdot)$ are the intensities for actual and clutter targets respectively. Thus it is sufficient to propagate the respective intensities for actual and clutter targets $v_k^{(1)}(\cdot)$ and $v_k^{(0)}(\cdot)$ alongside the hybrid cardinality distribution $\ddot{\rho}_k(\cdot)$. Moreover, the posterior intensity $v_k^{(0)}(\cdot)$ of the clutter generators is characterized by the posterior mean number of clutter generators $N_k^{(0)}$, since the detections or false alarms generated by clutter targets do not depend on the actual value of the clutter state c. The estimated posterior mean clutter rate is simply $\lambda_k = N_k^{(0)} p_{D,k}^{(0)}$ since the cardinality distribution of clutter is binomial. *Consequently, the CPHD filter for unknown clutter rate recursively propagates the following quantities:*

the posterior intensity $v_k^{(1)}(\cdot)$ of the actual target states, $N_k^{(0)}$ the posterior mean number of clutter generators, and $\ddot{\rho}_k(\cdot)$ the posterior cardinality distribution of all targets including actual and clutter.

Remark: The posterior cardinality $\ddot{\rho}_k(\ddot{n})$ of the hybrid state gives only information on the total number of actual and clutter targets. It is important to note that as a consequence of adopting a hybrid state space, the posterior mode $\hat{\ddot{N}}_k = \arg\max_{\ddot{n}} \ddot{\rho}_k(\ddot{n})$ cannot be used to estimate the number of actual targets, since this will include both actual targets and clutter generators. Instead, only the posterior mean $N_k^{(1)} = \left\langle 1, v_k^{(1)} \right\rangle$ can be used as an actual target number estimate.

Remark: The independence of the clutter returns from their clutter state values means that it is sufficient to specify the model for clutter completely in terms of: the mean number of clutter births $N_{\Gamma,k}^{(0)} = \left\langle 1, \gamma_k^{(0)} \right\rangle$ and constant probability of clutter target survival $p_{S,k}^{(0)}$, as well as the spatial likelihood $\tilde{\varkappa}_k(\cdot)$ and constant probability of clutter target detection $p_{D,k}^{(0)}$. It is not necessary to specify explicit forms for the transition density $f_{k|k-1}^{(0)}(\cdot|\cdot)$ and birth intensity $\gamma_k^{(0)}(\cdot)$.

The following results follow directly from substituting the hybrid state space model parameters into the conventional CPHD recursion, hence the proof is omitted. Notice that the use of clutter generators to model false alarms eliminates the calculation of the elementary symmetric functions and enforces the updated cardinality distribution to be zero for each argument up until $\ddot{n} = |Z_k|$. The resulting filter has a linear complexity in the number of measurements at the expense of less informative actual target cardinality.

Proposition 3 *If at time $k-1$, the posterior intensity for actual targets $v_{k-1}^{(1)}$, the posterior mean number of clutter generators $N_{k-1}^{(0)}$, and the posterior hybrid cardinality distribution $\ddot{\rho}_{k-1}$, are given, then their respective predictions to time k are given by*

$$v_{k|k-1}^{(1)}(x) = \gamma_k^{(1)}(x) + \int p_{S,k}^{(1)}(\zeta) f_{k|k-1}^{(1)}(x|\zeta) v_{k-1}^{(1)}(\zeta) d\zeta,$$
$$N_{k|k-1}^{(0)} = N_{\Gamma,k}^{(0)} + p_{S,k}^{(0)} N_{k-1}^{(0)},$$
$$\ddot{\rho}_{k|k-1}(\ddot{n}) = \sum_{j=0}^{\ddot{n}} \ddot{\rho}_{\Gamma,k}(\ddot{n}-j) \sum_{\ell=j}^{\infty} C_j^{\ell} \ddot{\rho}_{k-1}(\ell)(1-\phi)^{\ell-j}\phi^j$$

where $\phi = \frac{\left\langle v_{k-1}^{(1)}, p_{S,k}^{(1)} \right\rangle + N_{k-1}^{(0)} p_{S,k}^{(0)}}{\left\langle 1, v_{k-1}^{(1)} \right\rangle + N_{k-1}^{(0)}}$.

Proposition 4 *If at time k, the predicted intensity for actual targets $v_{k|k-1}^{(1)}$, the predicted mean number of clutter generators $N_{k|k-1}^{(0)}$, the predicted hybrid cardinality distribution $\ddot{\rho}_{k|k-1}$, are given, then their respective updates for a given sensor measurement set Z_k at time k are given by*

$$v_k^{(1)}(x)=\left[q_{D,k}^{(1)}(x)\frac{\frac{\left\langle \ddot{\Upsilon}_k^1[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}{\left\langle \ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}}{\left\langle 1,v_{k|k-1}^{(1)}\right\rangle+N_{k|k-1}^{(0)}} + \sum_{z\in Z_k}\frac{p_{D,k}^{(1)}(x)g_k(z|x)}{p_{D,k}^{(0)}N_{k|k-1}^{(0)}\tilde{\varkappa}_k(z)+\left\langle v_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right]v_{k|k-1}^{(1)}(x),$$

$$N_k^{(0)}=\left[q_{D,k}^{(0)}\frac{\frac{\left\langle \ddot{\Upsilon}_k^1[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}{\left\langle \ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1}Z_k],\ddot{\rho}_{k|k-1}\right\rangle}}{\left\langle 1,v_{k|k-1}^{(1)}\right\rangle+N_{k|k-1}^{(0)}} + \sum_{z\in Z_k}\frac{p_{D,k}^{(0)}\tilde{\varkappa}_k(z)}{p_{D,k}^{(0)}N_{k|k-1}^{(0)}\tilde{\varkappa}_k(z)+\left\langle v_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right]N_{k|k-1}^{(0)}$$

$$\ddot{\rho}_k(\ddot{n})=\begin{cases} 0 & \ddot{n}<|Z_k| \\ \frac{\ddot{\rho}_{k|k-1}(\ddot{n})\ddot{\Upsilon}_k^0[\ddot{v}_{k|k-1}Z_k](\ddot{n})}{\left\langle \ddot{\rho}_{k|k-1},\ddot{\Upsilon}_k^0\right\rangle} & \ddot{n}\geq |Z_k| \end{cases}$$

where

$$\ddot{\Upsilon}_k^u[\ddot{v}_{k|k-1}Z_k](\ddot{n}) = \begin{cases} 0 & \ddot{n}<|Z_k|+u \\ P_{|Z_k|+u}^{\ddot{n}}\Phi^{\ddot{n}-(|Z_k|+u)} & \ddot{n}\geq |Z_k|+u \end{cases},$$
$$\Phi = 1-\frac{\left\langle v_{k|k-1}^{(1)},p_{D,k}^{(1)}\right\rangle+N_{k|k-1}^{(0)}p_{D,k}^{(0)}}{\left\langle 1,v_{k|k-1}^{(1)}\right\rangle+N_{k|k-1}^{(0)}}$$

If the cardinalities of the RFS involved are Poisson distributed, the above propositions reduce to the following PHD recursion for unknown clutter rate:

$$v_{k|k-1}^{(1)}(x)=\gamma_k^{(1)}(x)+\left\langle v_{k-1}^{(1)}p_{S,k}^{(1)},f_{k|k-1}^{(1)}(x|\cdot)\right\rangle,$$
$$N_{k|k-1}^{(0)}=\left\langle 1,\gamma_k^{(0)}\right\rangle+p_{S,k}^{(0)}N_{k-1}^{(0)},$$
$$v_k^{(1)}(x)=\left[q_{D,k}^{(1)}(x)+\sum_{z\in Z_k}\frac{p_{D,k}^{(1)}(x)g_k(z|x)}{p_{D,k}^{(0)}N_{k|k-1}^{(0)}\tilde{\varkappa}_k(z)+\left\langle v_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right]v_{k|k-1}^{(1)}(x),$$
$$N_k^{(0)}=\left[q_{D,k}^{(0)}(x)+\sum_{z\in Z_k}\frac{p_{D,k}^{(0)}\tilde{\varkappa}_k(z)}{p_{D,k}^{(0)}N_{k|k-1}^{(0)}\tilde{\varkappa}_k(z)+\left\langle v_{k|k-1}^{(1)},p_{D,k}^{(1)}g_k(z|\cdot)\right\rangle}\right]N_{k|k-1}^{(0)}.$$

IV. CPHD Filtering with Unknown Detection Profile

The basic idea in this development is to augment the unknown detection probability into the single target state. This approach can accommodate (spatially) non-uniform detection profiles. Estimation of the augmented state of targets then yields information on the number of targets, as well as the individual kinematic states and the unknown detection probability for the particular target. In the following, we expand the CPHD recursion for the augmented state model.

A. Augmented State Space Model

Let $\mathcal{X}^{(\Delta)} = [0,1]$ denote the state space for the unknown detection probability. Define the augmented state space

$$\underline{\mathcal{X}} = \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)},$$

where $\times$ denotes a Cartesian product. The underscore notation is used throughout to denote a function or variable defined on the augmented state space, i.e. denote $\underline{x} = [x, a] \in \underline{\mathcal{X}}$ for an augmented state where $x \in \mathcal{X}^{(1)}$ for the kinematic state and

$a \in \mathcal{X}^{(\Delta)} = [0,1]$ for the augmented part. The integral of a function $\underline{f} : \underline{\mathcal{X}} \rightarrow \mathbb{R}$ is given by

$$\int_{\underline{\mathcal{X}}} \underline{f}(\underline{x}) d\underline{x} = \int_{\mathcal{X}^{(\Delta)}} \int_{\mathcal{X}^{(1)}} \underline{f}(x,a) dx da.$$

The multi-target transition and measurement models are essentially the same as in the conventional case, except that the single-target transition and measurement models are extended to accommodate the augmented state. For consistency in notations, the superscript $^{(1)}$ will continue to be used to denote functions or variables on the space of actual targets. Note that the conventional model applies for clutter, hence no clutter targets and no superscript $^{(0)}$ will be encountered in this subsection.

The single target survival probability and transition density for augmented states are simply

$$\underline{p}_{S,k}(\underline{x}) = \underline{p}_{S,k}(x,a) = p^{(1)}_{S,k}(x),$$
$$\underline{f}_{k|k-1}(\underline{x}|\underline{\zeta}) = \underline{f}_{k|k-1}(x,a|\zeta,\alpha) = f^{(1)}_{k|k-1}(x|\zeta) f^{(\Delta)}_{k|k-1}(a|\alpha).$$

Target births are given by an intensity $\underline{\gamma}^{(1)}_k(\cdot)$ for augmented states as well as corresponding usual cardinality distribution $\rho^{(1)}_{\Gamma,k}(\cdot)$. The single target detection probability and measurement likelihood for augmented states are

$$\underline{p}_{D,k}(\underline{x}) = \underline{p}_{D,k}(x,a) = a, \quad (14)$$
$$\underline{g}_k(z|\underline{x}) = \underline{g}_k(z|x,a) = g_k(z|x). \quad (15)$$

Clutter follows the conventional CPHD model given by Poisson false alarms with intensity function $\kappa_k(\cdot)$.

B. Recursion

The CPHD recursion for an augmented state model featuring unknown probability of detection can be obtained by substituting the single target motion and observation models for the augmented state into the conventional CPHD recursion given by Propositions 1 and 2. The resultant CPHD recursion propagates the posterior cardinality distribution $\rho_k(\cdot)$ and the posterior intensity function $\underline{v}^{(1)}_k(\cdot,\cdot)$ for the augmented state which now includes the unknown probability of detection. The following results are direct consequences of substituting the augmented state space model parameters into the conventional CPHD recursion. The new recursion retains the cubic complexity in measurements of the conventional CPHD recursion, since the conventional model for false alarms is used and consequently the computation of elementary symmetric functions is still required.

Proposition 5 *If at time $k-1$, the posterior intensity $\underline{v}^{(1)}_{k-1}$ and posterior cardinality distribution ρ_{k-1} are given, then the predicted cardinality distribution $\rho_{k|k-1}$ and predicted intensity $\underline{v}^{(1)}_{k|k-1}$ are given by*

$$\rho_{k|k-1}(n) = \sum_{j=0}^{n} \rho^{(1)}_{\Gamma,k}(n-j) \Pi_{k|k-1}[\underline{v}^{(1)}_{k-1}, \rho_{k-1}](j),$$
$$\underline{v}^{(1)}_{k|k-1}(x,a) = \underline{\gamma}^{(1)}_k(x,a) + \int\int_0^1 p_{S,k}(\zeta) f^{(\Delta)}_{k|k-1}(a|\alpha) f^{(1)}_{k|k-1}(x|\zeta) \underline{v}^{(1)}_{k-1}(\alpha,\zeta) d\alpha d\zeta,$$

where

$$\Pi_{k|k-1}[\underline{v},\rho](j) = \sum_{\ell=j}^{\infty} C^{\ell}_{j} \rho(\ell) \frac{\langle p_{S,k}, \underline{v}\rangle^{j} \langle 1-p_{S,k}, \underline{v}\rangle^{\ell-j}}{\langle 1, \underline{v}\rangle^{\ell}}.$$

Proposition 6 *If at time k, the predicted intensity $\underline{v}^{(1)}_{k|k-1}$ and predicted cardinality distribution $\rho_{k|k-1}$ are given, then for a given measurement set Z_k, the updated cardinality distribution ρ_k and updated intensity $\underline{v}^{(1)}_k$ are given by*

$$\rho_k(n) = \frac{\underline{\Upsilon}^0_k[\underline{v}^{(1)}_{k|k-1}; Z_k](n) \rho_{k|k-1}(n)}{\langle \underline{\Upsilon}^0_k[\underline{v}^{(1)}_{k|k-1}; Z_k], \rho_{k|k-1}\rangle},$$
$$\underline{v}^{(1)}_k(x,a) = \left[\begin{array}{l} (1-a) \frac{\langle \underline{\Upsilon}^1_k[\underline{v}^{(1)}_{k|k-1}; Z_k], \rho_{k|k-1}\rangle}{\langle \underline{\Upsilon}^0_k[\underline{v}^{(1)}_{k|k-1}; Z_k], \rho_{k|k-1}\rangle} + \\ \sum_{z\in Z_k} \underline{\psi}_{k,z}(x,a) \frac{\langle \underline{\Upsilon}^1_k[\underline{v}^{(1)}_{k|k-1}; Z_k - \{z\}], \rho_{k|k-1}\rangle}{\langle \underline{\Upsilon}^0_k[\underline{v}^{(1)}_{k|k-1}; Z_k], \rho_{k|k-1}\rangle} \end{array} \right] \times \underline{v}^{(1)}_{k|k-1}(x,a),$$

where

$$\underline{\Upsilon}^u_k[\underline{v}^{(1)}_{k|k-1}, Z_k](n) = \sum_{j=0}^{\min(|Z_k|,n)} (|Z_k|-j)! p_{K,k}(|Z_k|-j) P^n_{j+u} \times \frac{\langle 1-\underline{p}_{D,k}, \underline{v}^{(1)}_{k|k-1}\rangle^{n-(j+u)}}{\langle 1, \underline{v}^{(1)}_{k|k-1}\rangle^{n}} e_j\left(\Xi_k(\underline{v}^{(1)}_{k|k-1}, Z_k)\right),$$
$$\underline{p}_{D,k}(x,a) = a,$$
$$\underline{\psi}_{k,z}(x,a) = \frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)} g_k(z|x)\cdot a,$$
$$\Xi_k(\underline{v}^{(1)}_{k|k-1}, Z_k) = \left\{ \langle \underline{v}^{(1)}_{k|k-1}, \underline{\psi}_{k,z}\rangle : z \in Z_k \right\}.$$

Unlike the unknown clutter case, in CPHD filtering with unknown detection profile, estimates of the number of targets can be obtained as usual with the mode $N^{(1)}_k = \arg\max_n \rho_k(n)$ derived from the cardinality distribution or the mean $N^{(1)}_k = \left\langle 1, \underline{v}^{(1)}_k \right\rangle$ derived from the intensity function.

The following shows the reduction to the PHD recursion when the cardinalities of the RFS involved are Poisson distributed:

$$\underline{v}^{(1)}_{k|k-1}(x,a) = \underline{\gamma}_k(x,a) + \int\int_0^1 p_{S,k}(\zeta) f^{(\Delta)}_{k|k-1}(a|\alpha) f^{(1)}_{k|k-1}(x|\zeta) \underline{v}^{(1)}_{k-1}(\alpha,\zeta) d\alpha d\zeta,$$
$$\underline{v}^{(1)}_k(x,a) = (1-a)\underline{v}^{(1)}_{k|k-1}(x,a) + \sum_{z\in Z_k} \frac{a \cdot g_k(z|x) \underline{v}^{(1)}_{k|k-1}(x,a)}{\kappa_k(z) + \langle \underline{p}_{D,k} g_k(z|\cdot), \underline{v}^{(1)}_{k|k-1}\rangle}.$$

V. CPHD Filtering with Jointly Unknown Clutter Rate and Detection Profile

To accommodate jointly unknown clutter rate and detection profile, we simply combine the previous two techniques outlined in Sections III and IV. Consequently, the single target state space is both hybridized and augmented. Clutter or false alarms are modelled by an unknown and time varying number of clutter generators. The cardinality distribution of clutter is again binomial. Both actual and clutter targets have an augmented state variable, in addition to their kinematic states, to describe their unknown and possibly time varying probability of detection. The state space model, single target models and resultant recursions are stated formally as follows.

A. Hybrid and Augmented State Space Model

Define the hybrid and augmented state space

$$\underline{\ddot{\mathcal{X}}} = (\mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)}) \uplus (\mathcal{X}^{(0)} \times \mathcal{X}^{(\Delta)}).$$

Consistent with previous notations, the double dot is used throughout to denote a function or variable defined on the hybrid and augmented state space, and the underscore notation is used throughout to denote a function or variable defined on the augmented state space. Where a hybrid and augmented function or variable is encountered, a joint double dot and underscore notation is used, i.e. denote $\underline{\ddot{x}} \in \underline{\ddot{\mathcal{X}}}$ for a hybrid and augmented state as opposed to $\underline{x} = (x, a) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)}$ or $\underline{c} = (c, b) \in \mathcal{X}^{(0)} \times \mathcal{X}^{(\Delta)}$ for (augmented) actual or clutter states comprising the kinematic and augmented components respectively. The integral of a function $\underline{\ddot{f}} : \underline{\ddot{\mathcal{X}}} \rightarrow \mathbb{R}$ is given by

$$\int_{\underline{\ddot{\mathcal{X}}}} \underline{\ddot{f}}(\ddot{x}) d\ddot{x} = \int_{\mathcal{X}^{(\Delta)}} \int_{\mathcal{X}^{(1)}} \underline{\ddot{f}}(x, a) dx da + \int_{\mathcal{X}^{(\Delta)}} \int_{\mathcal{X}^{(0)}} \underline{\ddot{f}}(c, b) dc db.$$

As per previous developments, it is assumed throughout that actual targets and clutter objects are statistically independent. It is similarly assumed that all clutter generators are identical as far as the kinematic state is concerned, but not the augmented state, and hence it is possible to ignore any functional dependence on the kinematic part of the state of a clutter generator. The dynamical and measurement models are essentially the same as in the development of the filter for the hybrid state space, except that these models are now amended to include the incorporation of an augmented state space.

The joint probability of survival is defined piecewise:

$$\underline{\ddot{p}}_{S,k}(\underline{\ddot{x}}) = \begin{cases} p_{S,k}^{(1)}(x), & \underline{\ddot{x}} = (x, a) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)} \\ p_{S,k}^{(0)}, & \underline{\ddot{x}} = (c, b) \in \mathcal{X}^{(0)} \times \mathcal{X}^{(\Delta)} \end{cases},$$

The joint transition density is defined piecewise:

$$\underline{\ddot{f}}_{k|k-1}(\ddot{x}|\ddot{\zeta}) = f_{k|k-1}^{(1)}(x|\zeta) f_{k|k-1}^{(\Delta)}(a|\alpha),$$

if $\underline{\ddot{x}} = (x, a)$, and $\ddot{\zeta} = (\zeta, \alpha) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)}$, while

$$\underline{\ddot{f}}_{k|k-1}(\ddot{x}|\ddot{\zeta}) = f_{k|k-1}^{(0)}(c|\tau),$$

if $\underline{\ddot{x}} = (c, b)$, and $\ddot{\zeta} = (\tau, \upsilon) \in \mathcal{X}^{(0)} \times \mathcal{X}^{(\Delta)}$.

The joint birth intensity is defined piecewise, and the joint birth cardinality is given by a convolution

$$\underline{\ddot{\gamma}}_k(\underline{\ddot{x}}) = \begin{cases} \underline{\gamma}_k^{(1)}(x, a), & \underline{\ddot{x}} = (x, a) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)} \\ \underline{\gamma}_k^{(0)}(c, b), & \underline{\ddot{x}} = (c, b) \in \mathcal{X}^{(0)} \times \mathcal{X}^{(\Delta)} \end{cases},$$

$$\ddot{\rho}_{\Gamma,k}(\ddot{n}) = (\rho_{\Gamma,k}^{(1)} * \rho_{\Gamma,k}^{(0)})(\ddot{n}),$$

The joint probability of detection is defined piecewise:

$$\underline{\ddot{p}}_{D,k}(\underline{\ddot{x}}) = \begin{cases} a, & \underline{\ddot{x}} = (x, a) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)} \\ b, & \underline{\ddot{x}} = (c, b) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)} \end{cases},$$

The joint likelihood is defined piecewise:

$$\underline{\ddot{g}}_k(z|\underline{\ddot{x}}) = \begin{cases} g_k(z|x), & \underline{\ddot{x}} = (x, a) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)} \\ \tilde{\varkappa}_k(z), & \underline{\ddot{x}} = (c, b) \in \mathcal{X}^{(1)} \times \mathcal{X}^{(\Delta)} \end{cases}.$$

B. Recursion

The CPHD filter with jointly unknown clutter rate and detection probability propagates the posterior intensity $\underline{\ddot{v}}_k$ and posterior cardinality distribution $\ddot{\rho}_k(\cdot)$ of the hybrid and augmented state $\underline{\ddot{X}}_k$. Like the CPHD filter for unknown clutter rate, the intensity of the hybridized and augmented state $\underline{\ddot{v}}_k(\cdot)$ is decomposable into an intensity function of actual targets $\underline{v}_k^{(1)}(\cdot, \cdot)$ and clutter generators $\underline{v}_k^{(0)}(\cdot, \cdot)$. Note that the posterior intensity $\underline{v}_k^{(0)}(\cdot, \cdot)$ of the clutter generators is characterized by a single dependent variable $\underline{v}_k^{(0)}(\cdot)$, since the detections or false alarms generated by clutter targets do not depend on the actual value of the clutter state c. Again, the following propositions follow as direct result of combining the techniques presented in Sections III-B and IV-B.

Proposition 7 *If at time $k-1$, the posterior intensity for actual targets $\underline{v}_{k-1}^{(1)}$, the posterior intensity for clutter generators $\underline{v}_{k-1}^{(0)}$, and the posterior hybrid cardinality distribution $\ddot{\rho}_{k-1}$, are all given, then their respective predictions to time k are given by*

$$\underline{v}_{k|k-1}^{(1)}(x, a) = \underline{\gamma}_k^{(1)}(x, a) + \int\int_0^1 p_{S,k}^{(1)}(\zeta) f_{k|k-1}^{(\Delta)}(a|\alpha) f_{k|k-1}^{(1)}(x|\zeta) \underline{v}_{k-1}^{(1)}(\alpha, \zeta) d\alpha d\zeta,$$

$$\underline{v}_{k-1}^{(0)}(b) = \underline{\gamma}_k^{(0)}(b) + p_{S,k}^{(0)} \underline{v}_{k-1}^{(0)}(b),$$

$$\ddot{\rho}_{k|k-1}(\ddot{n}) = \sum_{j=0}^{\ddot{n}} \ddot{\rho}_{\Gamma,k}(\ddot{n} - j) \sum_{\ell=j}^{\infty} C_j^{\ell} \ddot{\rho}_{k-1}(\ell)(1 - \phi)^{\ell - j} \phi^j$$

where $\phi = \dfrac{\left\langle \underline{v}_{k-1}^{(1)}, p_{S,k}^{(1)} \right\rangle + \left\langle \underline{v}_{k-1}^{(0)}, p_{S,k}^{(0)} \right\rangle}{\left\langle 1, \underline{v}_{k-1}^{(1)} \right\rangle + \left\langle 1, \underline{v}_{k-1}^{(0)} \right\rangle}$.

Proposition 8 *If at time k, the predicted intensity for actual targets $\underline{v}_{k|k-1}^{(1)}$, the predicted intensity for clutter generators $\underline{v}_{k|k-1}^{(0)}$, the predicted hybrid cardinality distribution $\ddot{\rho}_{k|k-1}$, are all given, then their respective updates for a given sensor measurement set Z_k at time k are given by*

$$\underline{v}_k^{(1)}(x, a) = \left[\frac{(1-a) \left\langle \underline{\ddot{\Upsilon}}_k^1 [\underline{\ddot{v}}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle / \left\langle \underline{\ddot{\Upsilon}}_k^0 [\underline{\ddot{v}}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle}{\left\langle 1, \underline{v}_{k|k-1}^{(1)} \right\rangle + \left\langle 1, \underline{v}_{k|k-1}^{(0)} \right\rangle} + \sum_{z \in Z_k} \frac{a \cdot g_k(z|x)}{\left\langle \underline{v}_{k|k-1}^{(0)}, p_{D,k}^{(0)} \tilde{\varkappa}_k \right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right] \times \underline{v}_{k|k-1}^{(1)}(x, a),$$

$$\underline{v}_k^{(0)}(b) = \left[\frac{(1-b) \left\langle \underline{\ddot{\Upsilon}}_k^1 [\underline{\ddot{v}}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle / \left\langle \underline{\ddot{\Upsilon}}_k^0 [\underline{\ddot{v}}_{k|k-1} Z_k], \ddot{\rho}_{k|k-1} \right\rangle}{\left\langle 1, \underline{v}_{k|k-1}^{(1)} \right\rangle + \left\langle 1, \underline{v}_{k|k-1}^{(0)} \right\rangle} + \sum_{z \in Z_k} \frac{b \cdot \tilde{\varkappa}_k(z)}{\left\langle \underline{v}_{k|k-1}^{(0)}, p_{D,k}^{(0)} \tilde{\varkappa}_k \right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right]$$

$$\times\, \underline{v}_{k|k-1}^{(0)}(b),$$

$$\ddot{\rho}_k(\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| \\ \frac{\ddot{\rho}_{k|k-1}(\ddot{n})\ddot{\Upsilon}_k^0[\underline{v}_{k|k-1}Z_k](\ddot{n})}{\langle \ddot{\rho}_{k|k-1}, \ddot{\Upsilon}_k^0 \rangle} & \ddot{n} \geq |Z_k| \end{cases}$$

where

$$\ddot{\Upsilon}_k^u[\ddot{v}_{k|k-1}Z_k](\ddot{n}) = \begin{cases} 0 & \ddot{n} < |Z_k| + u \\ P_{|Z_k|+u}^{\ddot{n}} \Phi_{k|k-1}^{\ddot{n}-(|Z_k|+u)} & \ddot{n} \geq |Z_k| + u \end{cases},$$

$$\Phi = 1 - \frac{\left\langle \underline{v}_{k|k-1}^{(1)}, \underline{p}_{D,k}^{(1)} \right\rangle + \left\langle \underline{v}_{k|k-1}^{(0)}, \underline{p}_{D,k}^{(0)} \right\rangle}{\left\langle 1, \underline{v}_{k|k-1}^{(1)} \right\rangle + \left\langle 1, \underline{v}_{k|k-1}^{(0)} \right\rangle},$$

$$p_{D,k}^{(1)}(x,a) = a, \quad p_{D,k}^{(0)}(b) = b.$$

Remark: Estimates of the posterior cardinality for actual targets and clutter targets must be calculated as an expected a posterior estimate $\hat{N}_k^{(1)} = \left\langle v_k^{(1)}, 1 \right\rangle$. The expected a posteriori estimate of the mean clutter rate is $\hat{\lambda}_k = \left\langle \underline{v}_k^{(0)}, \underline{p}_{D,k}^{(0)} \right\rangle$.

The corresponding PHD recursion follows as a special case:

$$\underline{v}_{k|k-1}^{(1)}(x,a) = \underline{\gamma}_k^{(1)}(x,a) + \iint_0^1 p_{S,k}^{(1)}(\zeta) f_{k|k-1}^{(\Delta)}(a|\alpha) f_{k|k-1}^{(1)}(x|\zeta) \underline{v}_{k-1}^{(1)}(\alpha,\zeta) d\alpha d\zeta,$$

$$\underline{v}_{k-1}^{(0)}(b) = \underline{\gamma}_k^{(0)}(b) + p_{S,k}^{(0)} \underline{v}_{k-1}^{(0)}(b),$$

$$\underline{v}_k^{(1)}(x,a) = \left[1 - a + \sum_{z\in Z_k} \frac{a \cdot g_k(z|x)}{\left\langle \underline{v}_{k|k-1}^{(0)}, p_{D,k}^{(0)} \tilde{\varkappa}_k \right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right] \times \underline{v}_{k|k-1}^{(1)}(x,a),$$

$$\underline{v}_k^{(1)}(b) = \left[1 - b + \sum_{z\in Z_k} \frac{b \cdot \tilde{\varkappa}_k(z)}{\left\langle \underline{v}_{k|k-1}^{(0)}, p_{D,k}^{(0)} \tilde{\varkappa}_k \right\rangle + \left\langle \underline{v}_{k|k-1}^{(1)}, p_{D,k}^{(1)} g_k(z|\cdot) \right\rangle} \right] \times \underline{v}_{k|k-1}^{(1)}(b).$$

Details of analytic implementations for the proposed CPHD recursions based on Beta Gaussian mixtures are reported in [12]. In the following we demonstrate the proposed CPHD filter for jointly unknown clutter rate and detection profile with a numerical example using the implementation in [12].

VI. Numerical Studies

Consider a 10 target scenario on the region $[-1000, 1000]m \times [-1000, 1000]m$. Actual targets move with constant velocity as shown in Figure 1. The kinematic target state is a vector of planar position and velocity $x_k = [\ p_{x,k}, p_{y,k}, \dot{p}_{x,k}, \dot{p}_{y,k}\ ]^T$. Measurements are noisy vectors of planar position only $z_k = [\ z_{x,k}, z_{y,k}\ ]^T$. The single-target state space model is linear Gaussian with parameters

$$F_k = \begin{bmatrix} I_2 & \Delta I_2 \\ 0_2 & I_2 \end{bmatrix} \quad Q_k = \sigma_\nu^2 \begin{bmatrix} \frac{\Delta^4}{4} I_2 & \frac{\Delta^3}{2} I_2 \\ \frac{\Delta^3}{2} I_2 & \Delta^2 I_2 \end{bmatrix}$$
$$H_k = \begin{bmatrix} I_2 & 0_2 \end{bmatrix} \qquad R_k = \sigma_\varepsilon^2 I_2$$

where I_n and 0_n denote the $n \times n$ identity and zero matrices respectively, $\Delta = 1s$ is the sampling period, $\sigma_\nu = 5m/s^2$ is the standard deviation of the process noise, and $\sigma_\varepsilon = 10m$ is the standard deviation of the measurement noise.

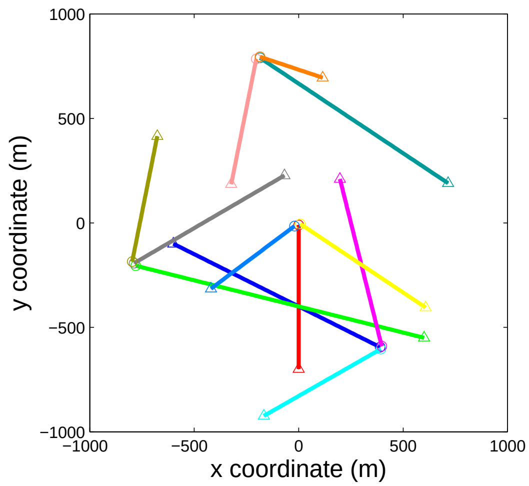

Figure 1. Trajectories in the xy plane. Start/Stop positions for each track are shown with ○/△.

The prediction for the augmented part of the state preserves the mean but increases the variance by 10%. The survival probability for actual targets is $p_{S,k}^{(1)} = 0.99$. The birth process for actual targets is a Poisson RFS with intensity $\underline{\gamma}_k^{(1)}(a,x) = \sum_{i=1}^4 w_\gamma \beta(a; u_\gamma, v_\gamma) \mathcal{N}(x; m_\gamma^{(i)}, P_\gamma)$,where $w_\gamma = 0.03$, $u_\gamma = 98$, $v_\gamma = 2$, $m_\gamma^{(1)} = [\ 0,0,0,0\ ]^T$, $m_\gamma^{(2)} = [\ 400,-600,0,0\ ]^T$, $m_\gamma^{(3)} = [\ -800,-200,0,0\ ]^T$, $m_\gamma^{(4)} = [\ -200,800,0,0\ ]^T$, and $P_\gamma = \text{diag}([\ 10,\ 10,\ 10,\ 10\ ]^T)^2$. Measurements are generated according to a constant detection probability $p_{D,k}^{(1)} = 0.98$ but this is not known to the filter and must be implicitly estimated at the location of each of the tracks.

The model for clutter generators given to the filter is that of births given by a mean rate of $N_{\Gamma,k}^{(0)} = 10$ while deaths are given by the survival probability of $p_{S,k}^{(0)} = 0.9$ and returns are given by detection probability $p_{D,k}^{(0)} = 0.5$. The density of clutter returns $\tilde{\varkappa}_k$ is presumed to be uniform on the measurement space. Clutter returns are generated according to a binomial cardinality with parameters $N_k^{(0)} = 20$ and $p_{D,k}^{(0)} = 0.5$ and uniform spatial probability density $1/V$ over the surveillance region where $V = 4 \times 10^6 m^2$ is the 'volume' of the surveillance region. The mean clutter rate is hence 10 points per scan and the intensity of clutter is $\lambda_k^{(0)} = N_k^{(0)} p_{D,k}^{(0)}/V = 5.00 \times 10^{-6} m^{-2}$. This information however is not known to the filter and must be dynamically estimated while filtering.

The filter is initialized with a zero intensity for actual targets (hence zero actual targets), and with a clutter rate equal to the total number of measurements received at the first time step minus the average birth and detection rate for actual targets.

Pruning and merging of Beta Gaussian components is performed at each time step using a weight threshold of $T' = 10^{-5}$, a merging threshold of $S' = 1\%$, and a maximum of $J_{max} = 3000$ Beta Gaussian components. The number of actual targets is estimated as the expected a posteriori cardinality and state estimates are extracted as the means of corresponding highest components of the posterior intensity and ignoring the augmented part of the state. The cardinality distribution is calculated to a maximum of $N_{\max} = 300$ terms.

The results for a single sample run are shown in Figure 2 with the target trajectories in x and y coordinates versus time plotted against those estimated from the CPHD filter. The Optimal Sub-Pattern Assignment (OSPA) multi-target miss-distance [17], with parameters $p = 1$ and $c = 300m$, is shown in Figure 3. It appears that the filter is able to initiate births, track locations, and identify deaths of targets reasonably well, but has some difficulty resolving closely spaced targets as seen from the increased OSPA error around the 40s mark. The latter might be expected given that the filter has to contend with two significant sources of model uncertainty.

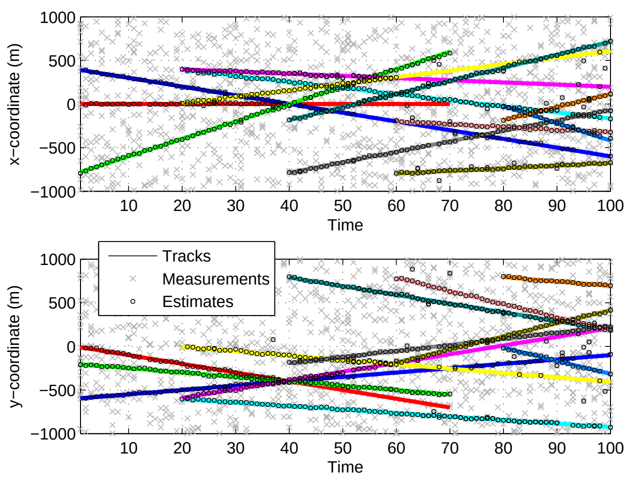

Figure 2. Beta-Gaussian CPHD filter for jointly unknown clutter rate and detection probability: filter estimates and true tracks in x and y coordinates versus time

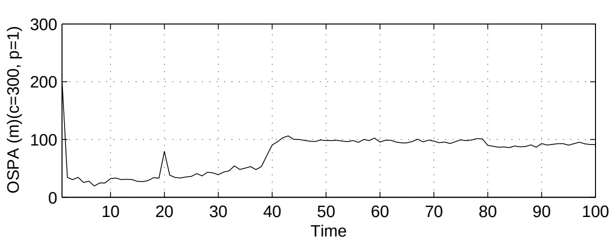

Figure 3. OSPA miss distance versus time for the CPHD filter for jointly unknown clutter rate and detection profile.

VII. Conclusion

We have shown how to use the CPHD (and PHD) filter in unknown clutter rate and/or detection profile. While the tracking performance cannot rival the ideal scenario where the clutter rate and detection profile are known a priori, numerical studies show that correct use of the CPHD (and PHD) filter can correct for discrepancies in these parameters and produces promising results. Future works will continue the development of CPHD, PHD filters to additionally accommodate other unknown model parameters. In addition, non-linearities in the model may be better handled by SMC implementations but another source of error arises in the extraction of the state estimate from the particle population, and the multi-Bernoulli approach of [18] may be preferable.

References

[1] R. Mahler, *Statistical Multisource-Multitarget Information Fusion*. Artech House, 2007.

[2] R. Mahler, "Multi-target Bayes filtering via first-order multi-target moments," *IEEE Trans. Aerospace and Electronic Systems*, vol. 39, no. 4, pp. 1152–1178, 2003.

[3] R. Mahler, "PHD filters of higher order in target number," *IEEE Trans. Aerospace & Electronic Systems*, Vol. 43, No. 3, pp. 1523–1543 2007.

[4] B.-N. Vo, S. Singh, and A. Doucet, "Sequential Monte Carlo methods for multi-target filtering with random finite sets," *in IEEE Trans. Aerospace and Electronic Systems*, vol. 41, no. 4, pp. 1224–1245, 2005.

[5] B.-N. Vo and W.-K. Ma, "The Gaussian mixture probability hypothesis density filter," *IEEE Trans. Signal Processing*, vol. 54, no. 11, pp. 4091–4104, Nov. 2006.

[6] B.-T. Vo, B.-N. Vo, and A. Cantoni, "Analytic implementations of the cardinalized probability hypothesis density filter," *IEEE Trans. Signal Processing*, vol. 55, no. 7, pp. 3553–3567, July. 2007.

[7] A. Pasha, B.-N. Vo, H. D. Tuan, and W. K. Ma, "A Gaussian mixture PHD filter for jump Markov system model," *IEEE Trans. Aerospace & Electronic Systems*, Volume 45, Issue 3, pp. 919-936, 2009.

[8] N. Whiteley, S. Singh, and S. Godsill, "Auxiliary Particle Implementation of Probability Hypothesis Density Filter", *in IEEE Trans. Aerospace and Electronic Systems*, vol. 46, no. 3, pp. 1437–1454, 2010.

[9] B. Ristic, D. Clark, and B.-N.Vo, "Improved SMC implementation of the PHD filter" *in Proc. 13th Annual Conf. Information Fusion*, Edinburgh, UK, 2010.

[10] J. Houssineau and D. Laneuville, "PHD filter with diffuse spatial prior on the birth process with applications to GM-PHD filter," *in Proc. 13th Annual Conf. Information Fusion*, Edinburgh, UK, 2010

[11] A. Elgammal, R. Duraiswami, D. Harwood, L. S. Davis, "Background and foreground modeling using nonparametric kernel density estimation for visual surveillance," *Proceedings of the IEEE* 90 (7). pp. 1151–1162, 2002.

[12] R. Mahler, B.-T. Vo, and B.-N. Vo, "CPHD filtering with unknown clutter rate and detection profile," *IEEE Trans. Signal Processing*, 2011 (to appear).

[13] R. Mahler, and A. El-Fallah, "CPHD filtering with unknown probability of detection," in I. Kadar (ed.), *Sign. Proc., Sensor Fusion, and Targ. Recogn*. XIX, SPIE Proc. Vol. 7697, 2010.

[14] R. Mahler, and A. El-Fallah, "CPHD and PHD filters for unknown backgrounds, III: Tractable multitarget filtering in dynamic clutter," in O. Drummond (ed.), *Sign. and Data Proc. of Small Targets 2010*, SPIE Proc. Vol. 7698, 2010.

[15] S. Singh, N. Whiteley, and S. Godsil, "An approximate likelihood method for estimating the static parameters in multi-target tracking models" *Technical Report CUED/F-INFENG/TR-606*, University of Cambridge, 2009.

[16] X. Chen, R. Tharmarasa, T. Kirubarajan, and M. Pelletier, "Integrated clutter estimation and target tracking using Poisson point process", *in Proc. SPIE,* O. Drummond Ed., Sign. and Data Proc. of Small Targets, Vol. 7445, 2009.

[17] D. Schuhmacher, B.-T. Vo, and B.-N. Vo, "A consistent metric for performance evaluation of multi-object filters," *IEEE Trans. Signal Processing*, Vol. 56, no. 8, pp. 3447–3457, Aug. 2008.

[18] B.-T. Vo, B.-N. Vo, and A. Cantoni, "The cardinality balanced multi-target multi-Bernoulli filter and its implementations," *IEEE Trans. Signal Processing*, Vol. 57, No. 2, pp. 409–423, Feb. 2009.