A Closed-Form Solution for the Probability Hypothesis Density Filter *

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Abstract – *The problem of dynamically estimating a time-varying set of targets can be cast as a filtering problem using the random finite set (or point process) framework. The probability hypothesis density (PHD) filter is a recursion that propagates the posterior intensity function–a 1st-order moment–of the random set of multiple targets in time. Like the Bayesian single-target filter, the PHD recursion also suffers from the curse of dimensionality. Although sequential Monte Carlo implementations have demonstrated the potential of the PHD filter, so far no closed-form solutions have yet been developed. In this paper, an analytic solution to the PHD recursion is proposed for linear Gaussian target dynamics with Gaussian births. This result is analogous to the Kalman recursion in Bayesian single-target filtering. Extension to nonlinear dynamics is also discussed.*

Keywords: Multi-target tracking, optimal filtering, point processes, random sets.

1 Introduction

The random finite set (RFS) framework has recently emerged as a promising approach to multi-target tracking. In the RFS formulation, the collection of individual targets is treated as a *set-valued state*, and the collection of individual observations is treated as a *set-valued observation*. Modeling set-valued states and set-valued observations as RFSs allows the problem of dynamically estimating multiple targets in the presence of clutter and association uncertainty to be cast in a Bayesian framework [1, 2, 3, 4]. This theoretically optimal approach to multi-target filtering is an elegant generalization of the single-target Bayes filter. Novel RFS-based filters such as the multi-target Bayes filter [1], the Probability Hypothesis Density (PHD) filter, [2] and their implementations [3, 4, 5, 6, 7] have generated substantial interests.

The focus of this paper is the PHD filter, a suboptimal but computationally tractable alternative to the multi-target Bayes filter. The implementation of the optimal multi-target Bayes filter is hindered by the need to evaluate combinatorial sums of multiple integrals, which has a prohibitively large number of combinations even for a medium number of targets. (Despite this drawback, the multi-target Bayes filter has been successful in applications with small number of targets; e.g., [8]). In PHD filtering, this combinatorial problem is circumvented by propagating only the first order moment of the multi-target posterior. However, the PHD recursion involves multi-dimensional integrals that generally do not have closed-form solutions. Current PHD filter implementations have been based on sequential Monte Carlo (SMC) [3, 4], a numerical integration method that can accurately approximate the PHD recursions given a sufficiently large number of particles [4] (see also [6, 7] for special cases of the SMC-PHD filter). Due to its ability to handle the time-varying number of nonlinear targets with relatively low complexity, innovative extensions and applications of the SMC-PHD filter soon followed [9, 10, 11, 12, 13]. However, so far no analytical solutions have yet been studied.

In this paper, we propose an analytic solution to the PHD recursion for linear Gaussian target dynamics with Gaussian births. It is shown that when the initial intensity is a Gaussian mixture, the posterior intensity at any subsequent time step is also a Gaussian mixture. Moreover, closed-form recursions for the weights, means, and covariances of the constituent Gaussian components are derived. The resulting filter progagates the Gaussian mixture posterior intensity in time as measurements arrive, in the same spirit as the Gaussian sum filter [14, 15]. In most cases of interest, the number of Gaussian components in the posterior intensity increases with time. However, this problem can be effectively mitigated by pruning the weak Gaussian components and keeping only the dominant ones at each time. Simulations are presented to demonstrate the multi-target tracking capability of the proposed Gaussian mixture PHD filter.

*This work is supported by an ARC Discovery grant

2 Background

This section presents an overview of multi-target filtering in the random finite set (or point process) framework. We begin with a review of single-target Bayesian filtering in Section 2.1. The multi-target estimation problem is then formulated in the Bayesian filtering framework by modeling multiple targets as random finite sets in Section 2.2. This provides sufficient background leading to Section 2.3, which describes the PHD filter. Readers who are familiar with random sets may go straight to to the PHD recursion in Section 2.3 without loss of continuity.

2.1 Single-target State Estimation

In many dynamic state estimation problems, the state is assumed to follow a Markov process on the state space $\mathcal{X} \subseteq \mathbb{R}^{n_x}$, with *transition density* $f_{k|k-1}(\cdot|\cdot)$, i.e. given a state x_{k-1} at time $k-1$, the probability density of a transition to the state x_k at time k is[1]

$$f_{k|k-1}(x_k|x_{k-1}). \tag{1}$$

This Markov process is partially observed in the observation space $\mathcal{Z} \subseteq \mathbb{R}^{n_z}$ as modeled by the *likelihood function* $g_k(\cdot|\cdot)$, i.e. given a state x_k at time k, the probability density of receiving the observation $z_k \in \mathcal{Z}$ is

$$g_k(z_k|x_k). \tag{2}$$

The probability density of the state x_k at time k given all observations $z_{1:k} = (z_1, \ldots, z_k)$ up to time k, denoted by

$$p_k(x_k|z_{1:k}), \tag{3}$$

is called the *posterior density* (or *filtering density*) at time k. From an initial density $p_0(\cdot)$, the posterior density at time k can be computed using the Bayes recursion

$$p_{k|k-1}(x_k|z_{1:k-1}) = \int f_{k|k-1}(x_k|\zeta)p_{k-1}(\zeta|z_{1:k-1})d\zeta, \tag{4}$$

$$p_k(x_k|z_{1:k}) = \frac{g_k(z_k|x_k)p_{k|k-1}(x_k|z_{1:k-1})}{\int g_k(z_k|\xi)p_{k|k-1}(\xi|z_{1:k-1})d\xi}. \tag{5}$$

All information (or uncertainty) about the state at time k is encapsulated in the posterior density $p_k(\cdot|z_{1:k})$, and estimates of the state at time k can be obtained using either the MMSE (Minimum Mean Squared Error) criterion or the MAP (Maximum A Posteriori) criterion. Subsequently, computing the posterior density at each time step forms the main task in Bayesian filtering.

[1]For notational simplicity, random variables and their realisations are not distinguished.

2.2 Random Finite Set in Multi-Target State Estimation

Consider a multi-target scenario. Let $x_{k,1}, \ldots, x_{k,M(k)}$ be the targets states and $z_{k,1}, \ldots, z_{k,N(k)}$ be the measurements at time k, where $M(k)$ and $N(k)$ are respectively the number of targets and measurements at time k. The key in the random finite set formulation is to treat the target set and measurement set

$$X_k = \{x_{k,1}, \ldots, x_{k,M(k)}\} \subset \mathcal{X}, \tag{6}$$

$$Z_k = \{z_{k,1}, \ldots, z_{k,N(k)}\} \subset \mathcal{Z}, \tag{7}$$

as the *multi-target state* and *multi-target observation* respectively. Uncertainty in a multi-target system is characterized by modeling the multi-target state X_k and multi-target measurement Z_k as *random finite sets* (RFS). A RFS X is simply a finite set-valued random variable, which can be described by a discrete probability distribution and a family of joint probability densities [16, 17]. The discrete distribution characterizes the cardinality of X, while for a given cardinality, an appropriate density characterizes the joint distribution of the elements of X.

In the following we describe a RFS model for the time evolution of the multi-target state which incorporates target motion, birth and death. For a given multi-target state X_{k-1} at time $k-1$, each $x_{k-1} \in X_{k-1}$ either continues to exist at time k with probability $e_{k|k-1}(x_{k-1})$ or dies with probability $1-e_{k|k-1}(x_{k-1})$.[2] Conditional on the target's existence at time k, the probability density of a transition from state x_{k-1} to state x_k is given by (1), i.e. $f_{k|k-1}(x_k|x_{k-1})$. Consequently, for a given (single-target) state $x_{k-1} \in X_{k-1}$, its behaviour at the next time step is modeled as the RFS $S_{k|k-1}(x_{k-1})$, which can take on either $\{x_k\}$ when the target survives, or $\emptyset$ when the target dies. Target birth is modeled by two components, namely spontaneous births (new targets that are independent of existing targets) and spawning (new targets generated from existing targets; e.g., missiles). Given a multi-target state X_{k-1} at time $k-1$, the multi-target state X_k at time k is given by the union of the surviving targets, the spawned targets and the spontaneous births i.e.

$$X_k = \left[\bigcup_{x \in X_{k-1}} S_{k|k-1}(x)\right] \cup \left[\bigcup_{x \in X_{k-1}} B_{k|k-1}(x)\right] \cup \Gamma_k. \tag{8}$$

where Γ_k denotes the RFS of spontaneous births, and $B_{k|k-1}(x_{k-1})$ denotes the RFS of targets spawned from $x_{k-1} \in X_{k-1}$. It is assumed that the RFSs constituting the union in (8) are independent of one another. Further discussions regarding the statistical assumptions on Γ_k and $B_{k|k-1}(x_{k-1})$ will be provided in Section 2.3.

The RFS measurement model is as follows. A given target $x_k \in X_k$ is either detected with probability $p_D(x_k)$ or

[2]Note that $e_{k|k-1}(x_{k-1})$ is a probability parameterized by x_{k-1}.

missed with probability $1-p_D(x_k)$.[3] Conditional on detection, the probability density of obtaining an observation z_k from x_k is given by (2), i.e. $g_k(z_k|x_k)$. Consequently, each state $x_k \in X_k$ generates a RFS $\Theta_k(x_k)$ that can take on either $\{z_k\}$ when the target is detected, or $\emptyset$ when the target is not detected. In addition to target originated measurements, the sensor receives a set K_k of false alarms or clutter. Thus, given a multi-target state X_k at time k, the multi-target measurement Z_k received at the sensor is formed by the union of target generated measurements and clutter, i.e.

$$Z_k = K_k \cup \left[\bigcup_{x \in X_k} \Theta_k(x) \right] \tag{9}$$

It is assumed that the RFS constituting the union in (9) are independent of one another. Further assumptions on the statistical properties of K_k are discussed in Section 2.3.

In a similar vein to the single-target dynamical model in (1) and (2), the randomness in the multi-target evolution and observation described by (8) and (9), are captured in the *multi-target transition density* $f_{k|k-1}(\cdot|\cdot)$ and *multi-target likelihood* $g_k(\cdot|\cdot)$ respectively[4] [2, 4].

Let $p_k(\cdot|Z_{1:k})$ denote the *multi-target posterior density*. Like the single-target Bayes recursion in (4) and (5), the multi-target posterior density can be recursively computed by

$$p_{k|k-1}(X_k|Z_{1:k-1}) = \int f_{k|k-1}(X_k|X) p_{k-1}(X|Z_{1:k-1}) \mu_s(dX) \tag{10}$$

$$p_k(X_k|Z_{1:k}) = \frac{g_k(Z_k|X_k) p_{k|k-1}(X_k|Z_{1:k-1})}{\int g_k(Z_k|X) p_{k|k-1}(X|Z_{1:k-1}) \mu_s(dX)}, \tag{11}$$

where μ_s is some reference measure on the space of finite subsets of $\mathcal{X}$ [4]. Although various applications of point process theory to multi-target tracking have been reported in the literature (e.g. [18, 19, 20]), Finite Set Statistics (FISST) [1, 2] is the first systematic approach to multi-target filtering that uses RFSs in such a Bayesian framework[5].

The recursion (10-11) involves multiple integrals on the space of finite sets, which are computationally intractable. The Probability Hypothesis Density (PHD) filter, presented next, is a more tractable alternative [2].

2.3 The PHD Filter

Instead of propagating the multi-target posterior density in time, the PHD filter propagates the posterior intensity, a first-order statistical moment associated with the multi-target posterior [2]. This strategy is reminiscent of the constant gain Kalman filter, which propagates the mean of the single-target state.

For a RFS X on $\mathcal{X}$ with probability distribution P, its first-order moment or intensity [17, 16] is a function $v : \mathcal{X} \to [0, \infty)$ such that for each region $S \subseteq \mathcal{X}$

$$\int_S v(x) dx = \int |X \cap S| \, P(dX), \tag{12}$$

where $|X|$ denotes the cardinality of a set X. In other words, the integral of v over any region S gives the expected number of elements of X that are in S. The intensity is commonly known in the tracking literature as the Probability Hypothesis Density (PHD).

The expected number of elements of X, $\hat{N} = \int v(x)dx$ can be used as an estimate for the number of targets. The local maxima of the intensity are points in $\mathcal{X}$ with the highest local concentration of expected number of targets and hence can be used to generate estimates for the elements of X. The simplest approach is to round $\hat{N}$ and choose the resulting number of highest peaks from the intensity.

A RFS X is *Poisson* if the cardinality distribution of X, $\Pr(|X| = n)$, is Poisson with mean $\hat{N}$, and for any finite cardinality, the elements x of X are i.i.d. according to the probability density $v(\cdot)/\hat{N}$ [17, 16]. A Poisson RFS is, thus, completely characterized by the intensity. For the multi-target problem described in the subsection 2.3, it is common to model the clutter RFS [K_k in (9)] and the birth RFSs [Γ_k and $B_{k|k-1}(x_{k-1})$ in (8)] as Poisson RFSs.

Let v_k and $v_{k|k-1}$ denote the respective intensities associated with the multi-target posterior density p_k and the multi-target predicted density $p_{k|k-1}$ in the recursion (10-11). Recall the multi-target dynamics and observation models from Section 2.2 with

$\gamma_k(\cdot)$	=	intensity of the birth RFS Γ_k at time k
$\beta_{k|k-1}(\cdot|\zeta)$	=	intensity of the RFS $B_{k|k-1}(\zeta)$ spawn at time k by a target previous state ζ
$e_{k|k-1}(\zeta)$	=	probability that a target still exist at time k given that its previous state is ζ
$p_D(x)$	=	probability of detection given a state x at time k
$\kappa_k(\cdot)$	=	intensity of the clutter RFS K_k at time k

and consider the following assumptions:

A.1. The clutter RFS K_k [in (9)] is independent of any other RFS.

A.2. The observation RFSs $\Theta(x_k)$ [in (9)] are mutually independent.

A.3. The predicted multi-target posterior density $p_{k|k-1}$ is completely characterized by the corresponding intensity $v_{k|k-1}$.

[3] Note that $p_D(x_k)$ is a probability parameterized by x_k.

[4] The same notation is used for multi-target and single-target densities. There is no danger of confusion since in the single-target case the arguments are vectors whereas in the multi-target case the arguments are finite sets.

[5] The FISST Bayes recursion takes on a slightly different form, since FISST is based on belief mass functionals, see [4] for more details

Under Assumptions A.1-A.3, it can be shown that the posterior intensity or PHD can be propagated in time through the following recursion [2]

$$v_{k|k-1}(x) = \int \phi_{k|k-1}(x,\zeta)v_{k-1}(\zeta)d\zeta + \gamma_k(x), \quad (13)$$

$$v_k(x) = [1 - p_D(x)]v_{k|k-1}(x) + \sum_{z\in Z_k} \frac{\psi_{k,z}(x)v_{k|k-1}(x)}{\kappa_k(z) + \int \psi_{k,z}(\xi)v_{k|k-1}(\xi)d\xi}, \quad (14)$$

where

$$\phi_{k|k-1}(x,\zeta) = e_{k|k-1}(\zeta)f_{k|k-1}(x|\zeta) + \beta_{k|k-1}(x|\zeta),$$
$$\psi_{k,z}(x) = p_D(x)g_k(z|x).$$

For notational convenience, the subscript k has been dropped from the state variable x since the index k is already indicated in the subscripts of $v_{k|k-1}$ and v_k. Since the posterior intensity is a function on the single-target state space $\mathcal{X}$, the PHD recursion (13-14) requires much less computational power than the multi-target recursion (10-11), which operates on the space of finite subsets of $\mathcal{X}$.

As mentioned in the introduction, generally the PHD recursion does not have an analytical form and is often handled by some numerical integration methods such as SMC [4]. The next section explores a scenario in which the PHD recursion can have a closed-form solution, namely the linear Gaussian scenario.

3 The Gaussian mixture PHD Filter

It is shown in this section that for a certain class of multi-target models, namely the linear Gaussian multi-target models, the PHD recursion (13-14) admits a closed-form solution. The linear Gaussian multi-target models are specified in Section 3.1, while the solution to the PHD recursion is presented in Section 3.2. In particular, it is shown that if the initial intensity is a Gaussian mixture, then the posterior intensity at any instant is also a Gaussian mixture. However, the number of components in these Gaussian mixtures increases with time. Section 3.3 examines the strategy of pruning to manage the growing number of Gaussians. In Section 3.4 we present extensions of the proposed closed-form method to mildly nonlinear dynamics.

3.1 Linear Gaussian Multi-Target Model

The assumptions for the *linear Gaussian multi-target model* are stated as follows:

A.4. Each target follows linear Gaussian dynamics i.e.

$$f_{k|k-1}(x|\zeta) = \mathcal{N}(x; F_{k-1}\zeta, Q_{k-1}), \quad (15)$$
$$g_k(z|x) = \mathcal{N}(z; H_k x, R_k), \quad (16)$$

where $\mathcal{N}(\cdot; m, P)$ denotes a Gaussian density with mean m and covariance P, F_{k-1} is the state transition matrix, Q_{k-1} is process noise covariance, H_k is the observation matrix, and R_k is the observation noise covariance.

A.5. The survival and detection probabilities are constants, i.e.

$$e_{k|k-1}(x) = e_k, \quad (17)$$
$$p_D(x) = p_D. \quad (18)$$

A.6. The intensities of the birth and spawn RFSs are Gaussian mixtures of the form

$$\gamma_k(x) = \sum_{i=1}^{J_k^{(\gamma)}} w_k^{(\gamma,i)}\mathcal{N}(x; m_k^{(\gamma,i)}, P_k^{(\gamma,i)}), \quad (19)$$

$$\beta_{k|k-1}(x|\zeta) = \sum_{j=1}^{J_k^{(\beta)}} w_k^{(\beta,j)}\mathcal{N}(x; \zeta + m_k^{(\beta,j)}, P_k^{(\beta,j)}). \quad (20)$$

where $w_k^{(\gamma,i)}$, $m_k^{(\gamma,i)}$, $P_k^{(\gamma,i)}$ are respectively the weight, mean and covariance of the ith Gaussian component of the birth intensity γ_k, and $w_k^{(\beta,j)}$, $\zeta + m_k^{(\beta,j)}$, $P_k^{(\beta,j)}$, are respectively the weight, mean and covariance of the ith Gaussian component of the spawning intensity $\beta_{k|k-1}(\cdot|\zeta)$ of target ζ.

Some remarks regarding the above assumptions are in order:

R.1. Assumptions A.4 and A.5 are commonly used in target tracking (e.g., in the joint probabilistic data association and multiple hypothesis tracking methods [21, 22]) to make the problem more tractable.

R.2. The Gaussian mixture Assumption A.6 on the spontaneous birth intensity provides the flexibility in modeling situations where target birth has high concentrations at several locations, such as airbases. A similar interpretation applies to the Gaussian mixture assumption on the target spawn intensity, except that a spawned target is modeled to be in close proximity to its parent.

3.2 The PHD Recursion

Consider the PHD recursion in (13) and (14) for the linear Gaussian multi-target model. The following two propositions show that the linear Gaussian PHD filter propagates a Gaussian mixture in time:

Proposition 1 *Suppose that Assumptions A.4-A.6 hold and that the posterior intensity at time $k-1$ is a Gaussian mixture of the form*

$$v_{k-1}(x) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)}\mathcal{N}(x; m_{k-1}^{(i)}, P_{k-1}^{(i)}). \quad (21)$$

Then, the predicted intensity at time k is also a Gaussian mixture, and is given by

$$v_{k|k-1}(x) = v_{k|k-1}^{(s)}(x) + v_{k|k-1}^{(\beta)}(x) + \gamma_k(x), \quad (22)$$

where

$$v^{(s)}_{k|k-1}(x) = e_k \sum_{i=1}^{J_{k-1}} w^{(i)}_{k-1}\mathcal{N}(x; m^{(i)}_{k|k-1}, P^{(i)}_{k|k-1}), \quad (23)$$

$$m^{(i)}_{k|k-1} = F_{k-1}m^{(i)}_{k-1}, \quad (24)$$

$$P^{(i)}_{k|k-1} = Q_{k-1} + F_{k-1}P^{(i)}_{k-1}F^T_{k-1}, \quad (25)$$

$$v^{(\beta)}_{k|k-1}(x) = \sum_{i=1}^{J_{k-1}} \sum_{j=1}^{J^{(\beta)}_k} w^{(i)}_{k-1}w^{(\beta,j)}_k \mathcal{N}(x; m^{(i,j)}_{k|k-1}, P^{(i,j)}_{k|k-1}), \quad (26)$$

$$m^{(i,j)}_{k|k-1} = m^{(i)}_{k-1} + m^{(\beta,j)}_k, \quad (27)$$

$$P^{(i,j)}_{k|k-1} = P^{(i)}_{k-1} + P^{(\beta,j)}_k. \quad (28)$$

Proposition 2 *Suppose that Assumptions A.4-A.6 hold and that the predicted intensity at time k is a Gaussian mixture of the form*

$$v_{k|k-1}(x) = \sum_{i=1}^{J_{k|k-1}} w^{(i)}_{k|k-1}\mathcal{N}(x; m^{(i)}_{k|k-1}, P^{(i)}_{k|k-1}). \quad (29)$$

Then, the posterior intensity at time k is also a Gaussian mixture, and is given by

$$v_k(x) = (1 - p_D)v_{k|k-1}(x) + \sum_{z\in Z_k} v^{(D)}_k(x; z) \quad (30)$$

where

$$v^{(D)}_k(x; z) = \sum_{i=1}^{J_{k|k-1}} \frac{p_D w^{(i)}_{k|k-1} q^{(i)}_k(z)\mathcal{N}(x; m^{(i)}_k(z), P^{(i)}_k)}{\kappa_k(z) + p_D \sum_{j=1}^{J_{k|k-1}} w^{(j)}_{k|k-1} q^{(j)}_k(z)} \quad (31)$$

$$q^{(j)}_k(z) = \mathcal{N}(z; H_k m^{(j)}_{k|k-1}, R_k + H_k P^{(j)}_{k|k-1}H^T_k), \quad (32)$$

$$m^{(i)}_k(z) = m^{(i)}_{k|k-1} + K^{(i)}_k(z - H_k m^{(i)}_{k|k-1}), \quad (33)$$

$$P^{(i)}_k = [I - K^{(i)}_k H_k]P^{(i)}_{k|k-1}, \quad (34)$$

$$K^{(i)}_k = P^{(i)}_{k|k-1}H^T_k(H_k P^{(i)}_{k|k-1}H^T_k + R_k)^{-1}. \quad (35)$$

Propositions 1 and 2 can be established by applying the following standard results for Gaussian functions (see for example [23] Section 3.8.)

Lemma 1 *Given F, d, Q, m, and P of appropriate dimensions, with Q and P positive definite,*

$$\int \mathcal{N}(x; F\zeta + d, Q)\mathcal{N}(\zeta; m, P)d\zeta = \mathcal{N}(x; Fm + d, Q + FPF^T) \quad (36)$$

Lemma 2 *Given H, R, m, and P of appropriate dimensions, with R and P positive definite,*

$$\mathcal{N}(z; Hx, R)\mathcal{N}(x; m, P) = q(z)\mathcal{N}(x; \tilde{m}, \tilde{P}) \quad (37)$$

where

$$q(z) = \mathcal{N}(z; H\tilde{m}, R + HPH^T) \quad (38)$$

$$\tilde{m} = m + K(z - Hm) \quad (39)$$

$$\tilde{P} = (I - KH)P \quad (40)$$

$$K = PH^T(HPH^T + R)^{-1} \quad (41)$$

In particular, Proposition 1 is established by substituting (15), (17), (19),(20), and (21), into the PHD prediction (13), and replacing integrals of the form (36) by appropriate Gaussians as given by Lemma 1. Similarly, Proposition 2 is established by substituting (16), (18), and (29) into the PHD update (14), and then replacing integrals of the form (36) and product of Gaussians of the form (37) by appropriate Gaussians as given by Lemmas 1 and 2 respectively.

It can be seen from Propositions 1 and 2 that at time k, the linear Gaussian PHD filter requires $(J_{k-1}(1 + J^{(\beta)}_k) + J^{(\gamma)}_k)(1 + |Z_k|)$ Gaussian components to construct $v_k(\cdot)$. Consequently, the linear Gaussian PHD recursion becomes increasingly expensive as k increases. This problem can be tackled (in an approximate manner) by pruning, as discussed in the next subsection.

3.3 Pruning for Efficient PHD Recursion

The purpose of pruning is to reduce the number of Gaussian components propagated to the next time step. Having computed the weights, means and covariances of the constituent components of the Gaussian mixture posterior density

$$v_k(x) = \sum_{i=1}^{J_k} w^{(i)}_k \mathcal{N}(x; m^{(i)}_k, P^{(i)}_k). \quad (42)$$

a good approximation can be obtained by truncating components that have weak weights $w^{(i)}_k$. Our experience with simulations (such as that in Section 4) indicated that many of the Gaussian components essentially contain negligible weights (say, of the order of 10^{-5}). Moreover, some of the Gaussian components are so close together that they could be accurately approximated by a single Gaussian. Hence, as an approximation these Gaussian components can be merged into one. These ideas lead to the simple heuristic pruning algorithm shown in Table 1.

3.4 Extension to Nonlinear Scenario

Suppose that the state dynamic and observation processes are nonlinear. Specifically, assumptions A.5 and A.6 still hold, but the state transition density and the likelihood

Table 1: A heuristic pruning algorithm for the Gaussian mixture PHD filter.

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given $\{w_k^{(i)}, m_k^{(i)}, P_k^{(i)}\}_{i=1}^{J_k}$, a truncation threshold T, and a merging threshold U.
Set $\ell = 1$, and $I = \{i = 1, \ldots, J_k | w_k^{(i)} > T\}$.
repeat

$j := \arg\max_{i \in I} w_k^{(i)}$.

$L := \{i \in I | (m_k^{(i)} - m_k^{(j)})^T (P_k^{(i)})^{-1} (m_k^{(i)} - m_k^{(j)}) \leq U\}$.

$\tilde{w}_k^{(\ell)} = \sum_{i \in L} w_k^{(i)}$.

$\tilde{m}_k^{(\ell)} = \frac{1}{\tilde{w}_k^{(\ell)}} \sum_{i \in L} w_k^{(i)} x_k^{(i)}$.

$\tilde{P}_k^{(\ell)} = \frac{1}{\tilde{w}_k^{(\ell)}} \sum_{i \in L} w_k^{(i)} (P_k^{(i)} + (\tilde{m}_k^{(\ell)} - m_k^{(i)})(\tilde{m}_k^{(\ell)} - m_k^{(i)})^T)$.

$\ell := \ell + 1$.

$I := I \backslash L$.

until $I = \emptyset$.
output $\{\tilde{w}_k^{(i)}, \tilde{m}_k^{(i)}, \tilde{P}_k^{(i)}\}_{i=1}^{\ell-1}$ as pruned Gaussian components.

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function respectively take on the following forms:

$$f_{k|k-1}(x|\zeta) = \mathcal{N}(x; \varphi_k(\zeta), Q_{k-1}) \tag{43}$$

$$g_k(z|x) = \mathcal{N}(z; h_k(x), R_k) \tag{44}$$

for some given nonlinear functions $\varphi_k : \mathcal{X} \to \mathcal{X}$ and $h_k : \mathcal{X} \to \mathcal{Z}$. Following the development in Section 3.2, one can show that the posterior intensity is still a weighted sum of some distribution functions, but those functions are no longer Gaussian due to the nonlinearity of φ_k and h_k. Nonetheless, the proposed Gaussian mixture implementation can be modified to accommodate nonlinear Gaussian dynamical models.

The key to implementing the non-linear Gaussian mixture PHD filter lies in finding non-linear non-Gaussian versions of Lemmas 1 and 2. An approximate version of Lemmas 1 and 2 for non-linear models can be obtained by linearizing the non-linear model as done in the Extended Kalman filter [24]. Alternatively, the unscented transform [25] can be used. Instead of linearizing the dynamical model, the unscented approach approximates each constituent component of the posterior intensity by a Gaussian as done in the unscented Kalman filter [25]. Using either one of these Gaussian approximation ideas, the proposed PHD Gaussian mixture implementation for the nonlinear Gaussian models follows in a straightforward manner.

4 Simulation

In this simulation example, target states are of the form $x_k = [\ p_{x,k}, \dot{p}_{x,k}, p_{y,k}, \dot{p}_{y,k}\ ]^T$, where $(p_{x,k}, p_{y,k})$ is the (x, y) coordinate of the target and $(\dot{p}_{x,k}, \dot{p}_{y,k})$ is the (x, y) velocity. The state transition is modeled by a nearly-constant velocity model [21], in which Eq. (15) (in A.4 in Section 3.1) are constructed by

$$F_k = \begin{bmatrix} \tilde{F} & 0 \\ 0 & \tilde{F} \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{45}$$

$$Q_k = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & \tilde{Q} \end{bmatrix}, \quad \tilde{Q} = \sigma_\varepsilon^2 \begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix} \tag{46}$$

where T is the sampling period, and σ_ε^2 is the variance of the state process noise. We set $T = 1s$ and $\sigma_\varepsilon = 5m/s^2$. Target generated measurements z_k contain the (x, y) target coordinates contaminated by noise. The constituent matrices for the likelihood function (in A.4 in Section 3.1) are given by

$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{47}$$

$$R_k = \sigma_w^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{48}$$

where σ_w^2 is the measurement noise variance. We set $\sigma_w^2 = 100m^2$. We assume that the spontaneous birth RFS is a Poisson RFS with intensity

$$\gamma_k(x_k) = 0.2\mathcal{N}(x_k; \bar{m}, \bar{P}) \tag{49}$$

$$\bar{m} = [\ 0,\ 3,\ 0,\ -3\ ]^T, \tag{50}$$

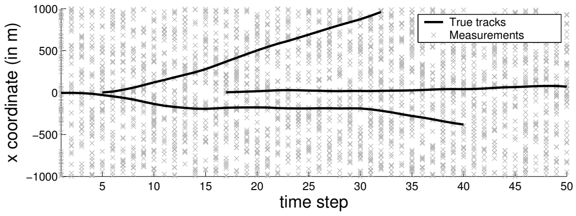
$$\bar{P} = \text{diag}([\ 10,\ 1,\ 10,\ 1\ ]^T). \tag{51}$$

In other words, the number of birth targets is Poisson distributed with an average rate of 0.2, and each birth state independently follows a Gaussian distribution $\mathcal{N}(x_k; \bar{m}, \bar{P})$. Moreover, we assume no target spawning. The target survival probability is set to $e_k = 0.99$. The probability of detection is $p_D = 0.98$. The clutter RFS is modeled by a Poisson RFS with intensity

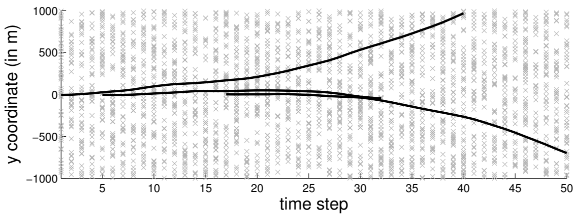
$$\kappa_k(z_k) = 50u(z_k) \tag{52}$$

where $u(.)$ is a uniform distribution function over $[-1000m, 1000m]^2$, which is the surveillance region of the tracking system. This implies that the average number of clutter points per unit area is 12.5 per $(km)^2$.

The measured data are plotted in Fig. 1. Fig. 2 shows the target position estimates of the linear Gaussian PHD filter. Note that the pruning procedure in Table 1 is used with the truncation threshold $T = 10^{-5}$ and the merging threshold $U = 4$. Moreover, the state estimates are obtained from the PHD filter by taking the means of the Gaussian components that have weights greater than $1/2$. We see that the Gaussian mixture PHD filter provides reasonably accurate tracking performance. The Gaussian mixture PHD filter does generate anomalous position estimates occasionally, but those false estimates do not seem to propagate with time.

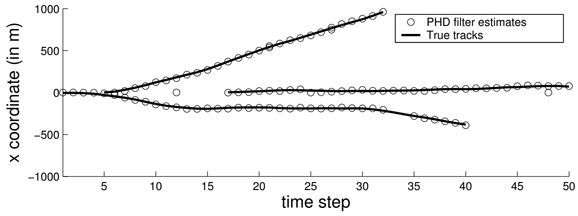


(a)

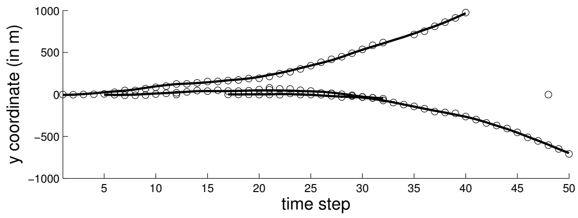


(b)

Figure 1: Measurement data and the true target positions.



(a)



(b)

Figure 2: Position estimates of the linear Gaussian PHD filter.

Fig. 3 compares the number of Gaussian components required by the exact Gaussian mixture PHD filter, and the number of Gaussian components used for the pruned (and thus approximate) Gaussian mixture PHD filter. The pruning parameters are same as those of the above simulation example. The figure indicates that pruning is a very effective procedure for implementing the Gaussian mixture PHD filter. We remark that the number of Gaussian terms for the pruned Gaussian mixture PHD filter is about 15 on average (the scale of the figure is not fine enough to show this).

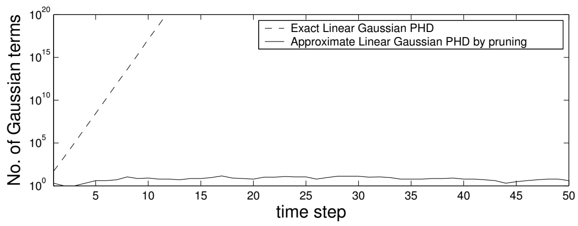


Figure 3: Numbers of Gaussian components required by linear Gaussian PHD filtering.

5 Conclusion and Discussion

This paper describes a closed-form solution for the PHD recursion for linear Gaussian multi-target models. We have shown that when the initial prior intensity is a Gaussian mixture, the posterior intensity at any time step is also a Gaussian mixture. In particular, we have derived closed-form recursions for the weights, means, and covariances of the constituent Gaussian components of the posterior intensity. A multi-target filter is proposed by combining the closed-form recursions with a simple pruning procedure to manage the growing number of Gaussian components. Moreover, this approach can be extended to mildly nonlinear models using approximation strategies from the extended Kalman filter, and the unscented Kalman filter. Simulations have demonstrated that the proposed approach is an attractive alternative to multi-target tracking with unknown time-varying number of targets.

Developing multi-target filters for sensor networks applications is a challenging future research direction. To perform tracking in sensor networks, it is critical to have efficient real-time algorithms that are capable of initiating and terminating tracks, and also account for detection uncertainties and false alarms. Since the PHD filter avoids solving the data association problem, the proposed algorithm satisfies all these requirements and is consquently promising for applications in sensor networks.

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