A Gaussian Mixture PHD Filter for Nonlinear Jump Markov Models

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***Abstract*— The probability hypothesis density (PHD) filter is an attractive approach to tracking an unknown, and time-varying number of targets in the presence of data association uncertainty, clutter, noise, and miss-detection. The PHD filter has a closed form solution under linear Gaussian assumptions on the target dynamics and births. However, the linear Gaussian multi-target model is not general enough to accommodate maneuvering targets, since these targets follow jump Markov system models. In this paper, we propose an analytic implementation of the PHD filter for jump Markov system (JMS) multi-target model. Our approach is based on a closed form solution to the PHD filter for linear Gaussian JMS multi-target model and the unscented transform. Using simulations, we demonstrate that the proposed PHD filtering algorithm is effective in tracking multiple maneuvering targets.**

I. Introduction

In a multi-target environment, the number of targets changes with time due to targets appearing, disappearing, and it is not known which target generated which measurement. Tracking multiple maneuvering targets involves jointly estimating the number of targets and their states at each time step. This problem is extremely difficult due to noise, clutter and uncertainties in target maneuvers, data association, and detection.

While non-maneuvering target motion can be described by a fixed model, a combination of motion models that characterize different maneuvers may be needed to describe the motion of a maneuvering target. The jump Markov system (JMS) model, or multiple models, approach has proven to be an effective tool for single maneuvering target tracking [1]. In this approach the target can switch between a set of models in a Markovian fashion. The jump Markov model approach can also be combined with traditional data association techniques such as joint probabilistic data association (JPDA) or multiple hypothesis tracking (MHT) to track multiple maneuvering targets. However, these data association-based approaches are computationally intensive in general.

Mahler's Probability Hypothesis Density (PHD) filter [2] circumvents the combinatorial computations that arise from data association while accommodating detection uncertainty, Poisson false alarms, target motions and time-varying number of targets. The generic sequential Monte Carlo implementation of the PHD filter [3] can accommodate any Markovian target dynamics including JMS models. However, the main drawbacks of the particle approach are the large number of particles, and the unreliability of clustering techniques for extracting state estimates [3], [4]. A closed form solution to the PHD recursion was proposed for linear Gaussian multi-target models in [4], [5] and generalized to handle linear Gaussian JMS (LGJMS) multi-target models in [6].

Although the LGJMS-PHD filter [6] shows great promise, many real world problems do not follow linear jump Markov models. Moreover, at present, there is no tractable analytical method for tracking multiple targets with nonlinear jump Markov dynamics. In this paper we present a simple extension of the LGJMS-PHD recursion to handle nonlinear jump Markov dynamics. This extension is based on an analytic approximation of the PHD recursion that combines the LGJMS-PHD filter [6] and the unscented transform [7]. The resulting multi-target filter sidesteps the data association problem, does not require gating, track initiation and termination, nor clustering for extracting state estimates.

II. Background

A. Jump Markov System

A jump Markov system (JMS) can be described by a set of parameterized state space models whose underlying parameters evolve with time according to a finite state Markov chain. Let $\xi_k \in \mathbb{R}^n$ and $z_k \in \mathbb{R}^m$ denote the kinematic state (e.g. target coordinates and velocity) and observation, respectively, at time k. Suppose that $r_k \in \mathcal{M}$ is the label of the model in effect at time k, where $\mathcal{M}$ denotes the (discrete) set of all model labels (also called modes). Then, the state dynamics and observation are described by the following state transition density and measurement likelihood:

$$\tilde{f}_{k|k-1}(\xi_k|\xi_{k-1}, r_k),$$
$$g_k(z_k|\xi_k, r_k).$$

In addition, the modes follow a discrete Markov chain with transition probability $t_{k|k-1}(r_k|r_{k-1})$ and the transition of the augmented state vector $x_k = [\xi_k^T, r_k]^T \in \mathcal{X} = \mathbb{R}^n \times \mathcal{M}$ is governed by

$$f_{k|k-1}(x_k|x_{k-1}) = \tilde{f}_{k|k-1}(\xi_k|\xi_{k-1}, r_k)t_{k|k-1}(r_k|r_{k-1}).$$

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A linear Gaussian JMS (LGJMS) is a JMS with linear Gaussian models, i.e. conditioned on mode r_k the state transition density and observation likelihood are given by

$$\tilde{f}_{k|k-1}(\xi_k|\xi_{k-1}, r_k) = \mathcal{N}(\xi_k; F_{k-1}(r_k)\xi_{k-1}, Q_{k-1}(r_k))$$
$$g_k(z_k|\xi_k, r_k) = \mathcal{N}(z_k; H_k(r_k)\xi_k, R_k(r_k)).$$

where $\mathcal{N}(\cdot; m, Q)$ denotes a Gaussian density with mean m and covariance Q, $F_{k-1}(r_k)$ and $H_k(r_k)$ denote the transition and observation matrices of model r_k respectively, $Q_{k-1}(r_k)$ and $R_k(r_k)$ denote covariance matrices of the process noise and measurement noise, respectively.

B. Random Finite Sets in Multi-target Tracking

In a multi-target scenario, suppose that $x_{k,1}, \ldots, x_{k,N(k)} \in \mathcal{X}$ are the augmented states at time k, where $N(k)$ denotes the number of targets. At the next time step, some of these targets may die, new targets may appear and the surviving targets evolve to their new states. At the sensor, $M(k)$ measurements $z_{k,1}, \ldots, z_{k,M(k)} \in \mathbb{R}^m$ are received at time k, some of which are due to targets while the rest are clutter. Note that only some of the existing targets are detected by the sensor, and that the corresponding measurements are indistinguishable from clutter. Hence, the orders in which the states, and the measurements are listed bear no significance.

Mahler's finite set statistics (FISST) approach provides an elegant Bayesian formulation of the multi-target filtering problem by treating the finite sets of targets and observations, at time k, as the *multi-target state* and *multi-target observation*, respectively [2]

$$\begin{aligned} X_k &= \{x_{k,1}, \ldots, x_{k,N(k)}\} \subset \mathcal{X}, \\ Z_k &= \{z_{k,1}, \ldots, z_{k,M(k)}\} \subset \mathbb{R}^m. \end{aligned}$$

To model uncertainty in multi-target states and observations, we appeal to the notion of a *random finite set* (RFS). An RFS on a state space $\mathcal{X}$ is simply a random variable taking values in the finite subsets of $\mathcal{X}$ [8]. The *intensity* of an RFS on $\mathcal{X}$ is a non-negative function v on $\mathcal{X}$ such that $v(x)$ is the instantaneous expected number of targets per unit volume at x. An RFS is *Poisson* if its cardinality distribution is Poisson with mean $N = \int v(x)dx$ and given a cardinality the elements of X are i.i.d. according to v/N. We refer the reader to [3], [4] for overviews on FISST and [2], [9] for comprehensive treatments.

C. The Probability Hypothesis Density Filter

The Probability Hypothesis Density (PHD) filter is a multi-target filter avoids any data association computations derived from the RFS framework [2]. The PHD filter propagates the posterior intensity of the RFS of targets in time, based on the following assumptions:

A.1 *Targets evolve in time and generate measurements independently of one another.*

A.2 *The clutter RFS is Poisson and is independent of the measurements.*

A.3 *The predicted multi-target RFS is Poisson.*

Assumptions A.1 and A.2 are quite common in many multi-target tracking algorithms. The additional assumption A.3 is a reasonable approximation in applications where interactions between targets are negligible [2].

The PHD propagation is a recursion consisting of a prediction step and an update step. Let $v_{k|k-1}$ and v_k denote the predicted intensity and posterior intensity at time k, respectively. Then the *PHD prediction* is given by

$$\begin{aligned} v_{k|k-1}(x) &= \int p_{S,k|k-1}(x') f_{k|k-1}(x|x') v_{k-1}(x') dx' \\ &\quad + \int \beta_{k|k-1}(x|x') v_{k-1}(x') dx' + \gamma_k(x), \end{aligned} \tag{1}$$

where it is understood that an integral with respect to a discrete variable means a sum, and

$f_{k|k-1}(\cdot|x')$ = probability density of a target at time k, given that its previous state is x',
$p_{S,k|k-1}(x')$ = probability that a target still exists at time k given that its previous state is x',
$\beta_{k|k-1}(\cdot|x')$ = intensity of the RFS of targets spawned at time k by a target with previous state x',
$\gamma_k(\cdot)$ = intensity of the birth RFS at time k.

On arrival of a new multi-target measurement, the posterior intensity v_k is computed from the predicted intensity $v_{k|k-1}$ via the *PHD update*:

$$\begin{aligned} v_k(x) &= [1 - p_{D,k}(x)] v_{k|k-1}(x) \\ &\quad + \sum_{z \in Z_k} \frac{p_{D,k}(x) g_k(z|x) v_{k|k-1}(x)}{\kappa_k(z) + \int p_{D,k}(x) g_k(z|x) v_{k|k-1}(x) dx}, \end{aligned} \tag{2}$$

where

Z_k = multi-target measurement at time k,
$g_k(\cdot|x)$ = single-target measurement likelihood at time k,
$p_{D,k}(x)$ = probability of detection given a state x at time k,
$\kappa_k(\cdot)$ = intensity of the clutter RFS at time k.

The PHD recursion is generally intractable due to the 'curse of dimensionality' in numerical integration. A generic sequential Monte Carlo (SMC) implementation was proposed in [3]. This so-called particle-PHD filter can accommodate targets with JMS dynamics, and has been used to track multiple maneuvering targets in [10], [11]. However, the main drawbacks of the particle approach are the large number of particles, and the unreliability of clustering techniques for extracting state estimates [3], [4]. The recently proposed Gaussian mixture PHD filter [4], [5] does not suffer from these drawbacks but is not general enough to handle targets with JMS dynamics.

III. Unscented Implementation of the PHD Filter

We present first a JMS multi-target model. For clarity in the presentation of our analytic implementation of the PHD filter for JMS multi-target model, we review the analytic solution to the PHD recursion for linear Gaussian JMS (LGJMS) multi-target model proposed in [6]. We then show

how the unscented transform is used to implement the PHD filter for nonlinear JMS multi-target model.

For notational convenience, Θ is used to denote the ordered pair of mean and covariance (m, P) of a Gaussian distribution, i.e.$\mathcal{N}(x;\Theta) = \mathcal{N}(x;m,P)$. Given a linear Gaussian model $z = Hx + v$, where v is Gaussian noise with mean d and covariance matrix R, we use the notation Ω to denote the ordered triplet of model parameters (H, R, d), and $\mathcal{L}(x, z; \Omega) = \mathcal{N}(z; Hx + d, R)$ to denote the probability density at z. We also define the operators Π and Ψ by

$$\Pi(\Omega,\Theta) = (Hm + d, R + HPH^T) \tag{3}$$

$$\Psi(z,\Omega,\Theta) = (m + K(z - d - Hm), (I - KH)P), \tag{4}$$

$$K = PH^T(HPH^T + R)^{-1}. \tag{5}$$

A. JMS multi-target model

In addition to assumptions A.1 - A.3, the *JMS multi-target model*, assumes:

A.4 *Each target follows a JMS model.*

A.5 *The probabilities of target survival and target detection are not functions of the kinematic state.*

A.5 *The intensities of birth and spawn RFS take the following forms:*

$$\gamma_k(\xi, r) = \tilde{\gamma}(\xi)\pi_k(r) \tag{6}$$

$$\beta_{k|k-1}(\xi, r|\xi', r') = \tilde{\beta}_{k|k-1}(\xi|\xi', r')\pi_{k|k-1}(r|r') \tag{7}$$

where $\tilde{\gamma}_k$ is the intensity of kinematic state births at time k, and $\pi_k(\cdot|\xi)$ is the probability distribution of the modes for a given birth with kinematic state ξ at time k, $\tilde{\beta}_{k|k-1}(\cdot|\xi', r')$ is the intensity of kinematic states spawned at time k from $[\xi'^T, r']^T$, and $\pi_{k|k-1}(\cdot|\xi, \xi', r')$ is the probability distribution of the mode for a given kinematic state ξ, spawned at time k from $[\xi'^T, r']^T$.

The JMS multi-target model is more general than those in standard multi-target tracking algorithms. Moreover, traditional multi-target filtering techniques are computationally intractable for a model of such generality. Most existing multiple maneuvering target tracking algorithms do not cater for births or spawnings.

B. Linear Gaussian JMS multi-target model

In the LGJMS multi-target model, each target follows a LGJMS model, i.e. the dynamics and measurement models for the kinematic state have the form:

$$\tilde{f}_{k|k-1}(\xi|\xi', r) = \mathcal{L}(\xi', \xi; \Omega_{f,k|k-1}(r)),$$

$$g_k(z|\xi, r) = \mathcal{L}(\xi, z; \Omega_{g,k}(r)),$$

where $\Omega_{f,k|k-1}(r) = (F_{f,k-1}(r), Q_{f,k-1}(r), 0)$ denotes the parameters of the linear target dynamics model conditioned on mode r, $\Omega_{g,k}(r) = (H_k(r), R_k(r), 0)$ denotes the parameters of the linear observation model conditioned on mode r. In particular, conditional on mode r, $F_{f,k-1}(r)$ is the state transition matrix, $Q_{f,k-1}(r)$ is the process noise covariance matrix, $H_k(r)$ is the measurement matrix and $R_k(r)$ is the measurement noise covariance matrix. Additionally, the birth and spawning intensities of the kinematic states are Gaussian mixtures:

$$\tilde{\gamma}_k(\xi) = \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)} \mathcal{N}(\xi; \Theta_{\gamma,k}^{(i)}),$$

$$\tilde{\beta}_{k|k-1}(\xi|\xi', r') = \sum_{j=1}^{J_{\beta,k|k-1}(r')} w_{\beta,k|k-1}^{(j)}(r') \mathcal{L}(\xi', \xi; \Omega_{\beta,k|k-1}^{(j)}(r')),$$

where $J_{\gamma,k}$, $\Theta_{\gamma,k}^{(i)} = (m_{\gamma,k}^{(i)}, Q_{\gamma,k}^{(i)})$, $w_{\gamma,k}^{(i)}$, $J_{\beta,k|k-1}(r')$, $\Omega_{\beta,k|k-1}^{(j)}(r') = (F_{\beta,k-1}^{(j)}(r'), Q_{\beta,k-1}^{(j)}(r'), d_{\beta,k-1}^{(j)}(r'))$, $w_{\beta,k|k-1}^{(j)}(r')$, are given model parameters.

C. Closed form PHD recursion

Proposition 1: *For a LGJMS multi-target model, if the posterior intensity v_{k-1} at time $k-1$ has the form*

$$v_{k-1}(\xi', r') = \sum_{i=1}^{J_{k-1}(r')} w_{k-1}^{(i)}(r') \mathcal{N}(\xi'; \Theta_{k-1}^{(i)}(r')). \tag{8}$$

Then the predicted intensity $v_{k|k-1}$ is given by

$$v_{k|k-1}(\xi, r) = \gamma_k(\xi, r) + v_{f,k|k-1}(\xi, r) + v_{\beta,k|k-1}(\xi, r), \tag{9}$$

where

$$v_{\beta,k|k-1}(\xi, r) = \sum_{r',i,j} w_{\beta,k|k-1}^{(i,j)}(r, r') \mathcal{N}(\xi; \Theta_{\beta,k|k-1}^{(i,j)}(r')), \tag{10}$$

$$w_{\beta,k|k-1}^{(i,j)}(r, r') = \pi_{k|k-1}(r|r') w_{\beta,k|k-1}^{(j)}(r') w_{k-1}^{(i)}(r'), \tag{11}$$

$$\Theta_{\beta,k|k-1}^{(i,j)}(r') = \Pi(\Omega_{\beta,k|k-1}^{(j)}(r'), \Theta_{k-1}^{(i)}(r')), \tag{12}$$

$$v_{f,k|k-1}(\xi, r) = \sum_{r',i} w_{f,k|k-1}^{(i)}(r, r') \mathcal{N}(\xi; \Theta_{f,k|k-1}^{(i)}(r, r')), \tag{13}$$

$$w_{f,k|k-1}^{(i)}(r, r') = p_{S,k|k-1}(r') t_{k|k-1}(r|r') w_{k-1}^{(i)}(r'), \tag{14}$$

$$\Theta_{f,k|k-1}^{(i)}(r, r') = \Pi(\Omega_{f,k|k-1}(r), \Theta_{k-1}^{(i)}(r')). \tag{15}$$

Proposition 2: *For a LGJMS multi-target model, if the predicted intensity $v_{k|k-1}$ has the form*

$$v_{k|k-1}(\xi, r) = \sum_{i=1}^{J_{k|k-1}(r)} w_{k|k-1}^{(i)}(r) \mathcal{N}(\xi; \Theta_{k|k-1}^{(i)}(r)). \tag{16}$$

Then the posterior intensity v_k is given by

$$v_k(\xi, r) = (1 - p_{D,k}(r)) v_{k|k-1}(\xi, r) + \sum_{z \in Z_k} v_{g,k}(\xi, r; z), \tag{17}$$

where

$$v_{g,k}(\xi, r; z) = \sum_i w_{g,k}^{(i)}(r; z) \mathcal{N}(\xi; \Theta_{g,k}^{(i)}(r; z)), \tag{18}$$

$$w_{g,k}^{(i)}(r; z) = \frac{p_{D,k}(r) w_{k|k-1}^{(i)}(r) q_{g,k}^{(i)}(r; z)}{\kappa_k(z) + \sum_{r,i} p_{D,k}(r) w_{k|k-1}^{(i)}(r) q_{g,k}^{(i)}(r; z)}, \tag{19}$$

$$q_{g,k}^{(i)}(r; z) = \mathcal{N}(z; \Pi(\Omega_{g,k}(r), \Theta_{k|k-1}^{(i)}(r))), \tag{20}$$

$$\Theta_{g,k}^{(i)}(r; z) = \Psi(z, \Omega_{g,k}(r), \Theta_{k|k-1}^{(i)}(r)). \tag{21}$$

Propositions 1 and 2 show how the intensities $v_{k|k-1}$ and v_k are analytically propagated in time under the l LGJMS multi-target model assumptions. The recursions for the means and covariances of $v_{f,k|k-1}$ and $v_{\beta,k|k-1}$ are the Kalman prediction and the recursive computations of the means and covariances of $v_{D,k}$ are the Kalman update. The PHD filter has a complexity of $\mathcal{O}(J_{k-1}|Z_k|)$ where J_{k-1} is the number of Gaussian components representing v_{k-1} for a fixed model r' at time $k-1$ and $|Z_k|$ denotes the number measurements at time k.

These propositions also indicate that the number of components of the predicted and posterior intensity increases with time, which can be a problem in implementation. However, this problem can be effectively handled by applying some simple pruning procedure [4], [5].

Given the posterior intensity v_k at time k

$$v_k(\xi,r) = \sum_{i=1}^{J_k(r)} w_k^{(i)}(r)\mathcal{N}(\xi;\Theta_k^{(i)}(r)),$$

the peaks of the intensity are points of highest local concentration of the expected number of targets. In order to extract the state of the targets from the posterior intensity at time k, an estimate of the number of targets $\hat{N}_k$ is needed. This number is simply $\sum_{i=1}^{J_k(r)} w_k^{(i)}(r)$ rounded to the nearest integer. The estimate of the multi-target state is the set of $\hat{N}_k$ ordered pairs of means and modes $(m_k^{(i)}(r), r)$ with the largest weights $w_k^{(i)}(r)$, $r \in \mathcal{M}, i = 1,\ldots,J_k(r)$.

Propositions 1 and 2 can be extended to exponential mixture probability of survival and probability of detection as in [4].

D. The Unscented JMS-PHD filter

We consider the following form of single target dynamical and measurement model for a given mode r

$$\xi_k = \varphi_{f,k-1}(\xi_{k-1},\nu_{f,k-1},r), \quad (22)$$
$$z_k = h_k(\xi_k,\varepsilon_k,r), \quad (23)$$

where $\varphi_{f,k-1}$ and h_k are the non-linear mappings, and $\nu_{f,k-1}$ and ε_k are independent zero-mean Gaussian noise processes with covariance matrices $Q_{f,k-1}(r)$ and $R_k(r)$ respectively. In addition, the spawning intensity has the form

$$\tilde{\beta}_{k|k-1}(\xi|\xi',r') = \sum_{j=1}^{J_{\beta,k|k-1}(r')} w_{\beta,k|k-1}^{(j)}(r')\mathcal{N}(\xi;\varphi_{\beta,k-1}(\xi',r'),Q_{\beta,k-1}^{(j)}(r')), \quad (24)$$

where $J_{\beta,k|k-1}(r')$, $w_{\beta,k|k-1}^{(j)}(r')$, $Q_{\beta,k-1}^{(j)}(r')$ are given model parameters, and $\varphi_{\beta,k-1}^{(j)}(\cdot,r')$ is a nonlinear function on the kinematic state space $\mathbb{R}^n$.

In single-target filtering, analytic approximations of the nonlinear Bayes filter include the extended Kalman (EK) filter and the unscented Kalman (UK) filter [7]. The EK filter approximates the posterior density by a Gaussian, which is propagated in time by applying the Kalman recursions to local linearizations of the (nonlinear) mappings φ_k and h_k. The UK filter also approximates the posterior density by a Gaussian, but instead of using the linearized model, it computes the Gaussian approximation of the posterior density at the next time step using the unscented transform. Details of the EK and UK filters are given in [12] and [7], respectively.

Following the development in Section III-C, it can be shown that the posterior intensity of the multi-target state propagated by the PHD recursions (1)-(2) is a weighted sum of various functions of ξ, many of which are non-Gaussian. In the same vein as UK filter, we can approximate each of these non-Gaussian constituent functions by a Gaussian.

1) Prediction: To begin, suppose that the posterior intensity at time $k-1$ is approximated by

$$v_{k-1}(\xi',r') \approx \sum_{i=1}^{J_{k-1}(r')} w_{k-1}^{(i)}(r')\mathcal{N}(\xi';\Theta_{k-1}^{(i)}(r')),$$

and that each mixture component $\mathcal{N}(\cdot;\Theta_{k-1}^{(i)}(r'))$ of the posterior intensity, has an associated set of sigma points[1] $\{\xi_{k-1}^{(\ell)}(i,r')\}_{\ell=0}^{L}$, with mean $m_{k-1}^{(i)}(r')$ and covariance $P_{k-1}^{(i)}(r')$.

The predicted intensity still has the same form as (9). Consider first, $v_{f,k|k-1}$, the motion component of the predicted intensity in (9). For each mode r of the state transition generate a set of sigma points $\{\nu_{f,k-1}^{(\ell)}(r)\}_{\ell=0}^{L}$, with mean 0 and covariance $Q_{f,k-1}(r)$. Then, following the unscented Kalman prediction [7] we can approximate $v_{f,k|k-1}$ by substituting into (13)

$$\Theta_{f,k|k-1}^{(i)}(r,r') = (m_{k|k-1}^{(i)}(r,r'), P_{k|k-1}^{(i)}(r,r')), \quad (25)$$

where

$$m_{k|k-1}^{(i)}(r,r') = \frac{1}{1+L}\sum_{\ell=0}^{L}\xi_{k|k-1}^{(\ell)}(i,r,r'), \quad (26)$$

$$\xi_{k|k-1}^{(\ell)}(i,r,r') = \varphi_{f,k-1}(\xi_{k-1}^{(\ell)}(i,r'),\nu_{f,k-1}^{(\ell)}(i,r),r), \quad (27)$$

$$P_{k|k-1}^{(i)}(r,r') = \frac{1}{1+L}\sum_{\ell=0}^{L}\tilde{\xi}_{k|k-1}^{(\ell)}(i,r,r')\tilde{\xi}_{k|k-1}^{(\ell)}(i,r,r')^T, \quad (28)$$

$$\tilde{\xi}_{k|k-1}^{(\ell)}(i,r,r') = \xi_{k|k-1}^{(\ell)}(i,r,r') - m_{k|k-1}^{(i)}(r,r'), \quad (29)$$

Similarly, to approximate $v_{\beta,k|k-1}$, the spawning component of (9), we generate a set of sigma points $\{\nu_{\beta,k-1}^{(\ell)}(j,r')\}_{\ell=0}^{L}$, with mean 0 and covariance $Q_{\beta,k-1}^{(j)}(r')$, for each (j,r') and substitute into (10)

$$\Theta_{\beta,k|k-1}^{(i,j)}(r') = (m_{k|k-1}^{(i,j)}(r'), P_{k|k-1}^{(i,j)}(r')), \quad (30)$$

where

$$m_{k|k-1}^{(i,j)}(r') = \frac{1}{1+L}\sum_{\ell=0}^{L}\xi_{k|k-1}^{(\ell)}(i,j,r'), \quad (31)$$

$$\xi_{k|k-1}^{(\ell)}(i,j,r') = \varphi_{\beta,k-1}(\xi_{k-1}^{(\ell)}(i,r')) + \nu_{\beta,k-1}^{(\ell)}(j,r'), \quad (32)$$

$$P_{k|k-1}^{(i,j)}(r') = \frac{1}{1+L}\sum_{\ell=0}^{L}\tilde{\xi}_{k|k-1}^{(\ell)}(i,j,r')\tilde{\xi}_{k|k-1}^{(\ell)}(i,j,r')^T, \quad (33)$$

$$\tilde{\xi}_{k|k-1}^{(\ell)}(i,j,r') = \xi_{k|k-1}^{(\ell)}(i,j,r') - m_{k|k-1}^{(i,j)}(r'), \quad (34)$$

[1] see [7] and references therein for details on sigma points.

To complete the prediction step, we approximate the spontaneous birth intensity γ_k in (9) by a Gaussian mixture

$$\sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)}(r)\mathcal{N}(\xi;\Theta_{\gamma,k}^{(i)}).$$

2) Update: For the update step, suppose that the predicted intensity is approximated by

$$v_{k|k-1}(\xi,r) \approx \sum_{i=1}^{J_{k|k-1}(r)} w_{k|k-1}^{(i)}(r)\mathcal{N}(\xi;\Theta_{k|k-1}^{(i)}(r)),$$

and that each component has an associated set of sigma points $\{\xi_{k|k-1}^{(\ell)}(i,r)\}_{\ell=0}^{L}$, with mean $m_{k|k-1}^{(i)}(r)$ and covariance $P_{k|k-1}^{(i)}(r)$.

The updated posterior intensity still has the same form as (17). The first term on the right hand side of (17) is just a scaled version of $v_{k|k-1}$. To approximate $v_{g,k}(\xi,r;z)$ in (17), we generate a set of sigma points $\{\varepsilon_k^{(\ell)}(r)\}_{\ell=0}^{L}$, with mean 0 and covariance $R_k(r)$ for each mode r. Then, following the unscented Kalman update [7] we can approximate $v_{g,k}(\xi,r;z)$ by substituting into (18)-(19)

$$q_{g,k}^{(i)}(r;z) = \mathcal{N}(z;\eta_{k|k-1}^{(i)}(r),S_k^{(i)}(r)), \quad (35)$$

$$\Theta_{g,k}^{(i)}(r;z) = (m_k^{(i)}(r;z),P_k^{(i)}(r)), \quad (36)$$

where

$$\eta_{k|k-1}^{(i)}(r) = \frac{1}{1+L}\sum_{\ell=0}^{L} z_{k|k-1}^{(\ell)}(i,r), \quad (37)$$

$$z_{k|k-1}^{(\ell)}(i,r) = h_k(\xi_{k|k-1}^{(\ell)}(i,r),\epsilon_k^{(\ell)}(r),r), \quad (38)$$

$$S_k^{(i)}(r) = \frac{1}{1+L}\sum_{\ell=0}^{L} \tilde{z}_{k|k-1}^{(\ell)}(i,r)\tilde{z}_{k|k-1}^{(\ell)}(i,r)^T, \quad (39)$$

$$\tilde{z}_{k|k-1}^{(\ell)}(i,r) = z_{k|k-1}^{(\ell)}(i,r) - \eta_{k|k-1}^{(i)}(r), \quad (40)$$

$$m_k^{(i)}(r;z) = m_{k|k-1}^{(i)}(r) - K_k^{(i)}(r)(z-\eta_{k|k-1}^{(i)}(r)), \quad (41)$$

$$P_k^{(i)}(r) = P_{k|k-1}^{(i)}(r) - K_k^{(i)}(r)G_k^{(i)}(r)^T, \quad (42)$$

$$K_k^{(i)}(r) = G_k^{(i)}(r)[S_k^{(i)}(r)]^{-1}, \quad (43)$$

$$G_k^{(i)}(r) = \frac{1}{1+L}\sum_{\ell=0}^{L} \tilde{\xi}_{k|k-1}^{(\ell)}(i,r)\tilde{\xi}_{k|k-1}^{(\ell)}(i,r)^T, \quad (44)$$

$$\tilde{\xi}_{k|k-1}^{(\ell)}(i,r) = \xi_{k|k-1}^{(\ell)}(i,r) - m_{k|k-1}^{(i)}(r). \quad (45)$$

A similar implementation can be done with the EK approach. In this case we approximate the nonlinear mappings $\varphi_{f,k-1}(\cdot,\nu_{f,k-1},r)$, $h_k(\cdot,\varepsilon_k,r)$ and $\varphi_{\beta,k-1}(\cdot,\nu_{\beta,k-1},r')$ by their respective derivatives and apply Propositions 1 and 2. However, calculating these derivatives may be tedious and error-prone. The unscented approach, on the other hand, does not suffer from these restrictions and can even be applied to models with discontinuities. For these reasons we omit the EK implementation in this paper.

IV. Simulation Results

Consider a two-dimensional scenario where aircrafts appear in a surveillance region at different locations and times. The speed of the aircrafts is in the range Mach [0.45, 0.6]. The kinematic state $\xi = [\ p_x,\, \dot{p}_x,\, p_y,\, \dot{p}_y,\, \Omega\ ]^T$ of each aircraft consists of position (p_x,p_y) in the horizontal plane, velocity $(\dot{p}_x,\dot{p}_y)$ and turn rate Ω. Model $r=1$ is the standard linear constant velocity (CV) model as given in [1]. Model $r=2$ is the non-linear co-ordinated turn (CT) model [1] with an unknown turn rate Ω given by

$$F_{k-1}(\Omega,r=2) = \begin{bmatrix} A_2 & -\tilde{A}_2 & 0 \\ \tilde{A}_2 & A_2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q_{k-1}(r=2) = \begin{bmatrix} \Sigma_2 & 0_2 & 0 \\ 0_2 & \Sigma_2 & 0 \\ 0 & 0 & \tilde{\Sigma}_2 \end{bmatrix},$$

with

$$A_2 = \begin{bmatrix} 1 & \frac{\sin\Omega T}{\Omega} \\ 0 & \cos\Omega T \end{bmatrix},\ \tilde{A}_2 = \begin{bmatrix} 0 & \frac{1-\cos\Omega T}{\Omega} \\ 0 & \sin\Omega T \end{bmatrix},$$

$$\Sigma_2 = \sigma_{v_2}^2 \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix},\ \tilde{\Sigma}_2 = T\tilde{\sigma}_{v_2}^2,$$

where $\sigma_{v_2} = 10\,m\,s^{-2}$ and $\tilde{\sigma}_{v_2} = 0.5^\circ\,s^{-2}$ are the standard deviations of the noise for the linear and turn portions of the kinematic state during a level turn, respectively. The transition probabilities between the modes are given by

$$\left[t_{k|k-1}(r|r')\right] = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}.$$

The aircrafts are observed by a bearing and range-only sensor in a region $[-\pi,\pi]\,rad \times [0, 6\times 10^4]\,m$

$$z_k = \begin{bmatrix} \arctan(p_{x,k}/p_{y,k}) \\ \sqrt{p_{x,k}^2 + p_{y,k}^2} \end{bmatrix} + \varepsilon_k,$$

where $\varepsilon_k \sim \mathcal{N}(\cdot;0,R_k)$ with $R_k = diag([\,\sigma_\theta^2,\sigma_\rho^2\,])$, $\sigma_\theta = 2\times(\pi/180)\,rad\,s^{-1}$ and $\sigma_\rho = 20\,m$. The interval between the sensor measurements is $T=5s$. The probability that an aircraft survives at the next time step is taken as $p_{S,k|k-1}(r') = 0.99$ and the probability that an aircraft is detected is $p_{D,k}(r) = 0.98$. Clutter is modelled as a Poisson RFS with intensity $\kappa_k(z) = \lambda_c V \mathcal{U}(z)$, where $\mathcal{U}(\cdot)$ denotes a uniform density over the surveillance region, V the *volume* of the surveillance region and $\lambda_c = 4.167\times10^{-4}/\pi$ denotes the average number of clutter returns per unit volume.

For simplicity target spawning is not considered. Consider a scenario where the surveillance region includes the five airport locations at $(-20,-20)km$, $(10,20)km$, $(30,-10)km$, $(-30,20)km$ and $(-20,40)km$. The spontaneous birth RFS is Poisson with intensity

$$\gamma_k(x) = 0.1\pi_k(r)\big[\mathcal{N}(\xi;m_\gamma^{(1)},P_\gamma)+\mathcal{N}(\xi;m_\gamma^{(2)},P_\gamma)+ \mathcal{N}(\xi;m_\gamma^{(3)},P_\gamma)+\mathcal{N}(\xi;m_\gamma^{(4)},P_\gamma)+\mathcal{N}(\xi;m_\gamma^{(5)},P_\gamma)\big],$$

with

$$
\begin{aligned}
m_\gamma^{(1)} &= \left[\ -2\times 10^4,\ \ 0,\ \ -2\times 10^4,\ \ 0,\ \ 0\ \right]^T, \\
m_\gamma^{(2)} &= \left[\ 1\times 10^4,\ \ 0,\ \ 2\times 10^4,\ \ 0,\ \ 0\ \right]^T, \\
m_\gamma^{(3)} &= \left[\ 3\times 10^4,\ \ 0,\ \ -1\times 10^4,\ \ 0,\ \ 0\ \right]^T, \\
m_\gamma^{(4)} &= \left[\ -3\times 10^4,\ \ 0,\ \ 2\times 10^4,\ \ 0,\ \ 0\ \right]^T, \\
m_\gamma^{(5)} &= \left[\ -2\times 10^4,\ \ 0,\ \ 4\times 10^4,\ \ 0,\ \ 0\ \right]^T, \\
P_\gamma &= diag\Big(\left[\ 10^3,\ \ 200,\ \ 10^3,\ \ 200,\ \ 0\ \right]\Big),
\end{aligned}
$$

and the distribution of the models at birth is taken as

$$[\pi_k(r)] = \ [0.8 \quad 0.2]\ .$$

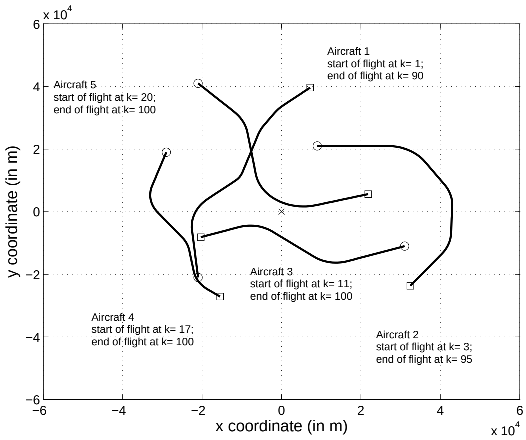

Fig. 1. Target trajectories. '○'– locations of target births; '□'– locations of target deaths ('×'– location of sensor).

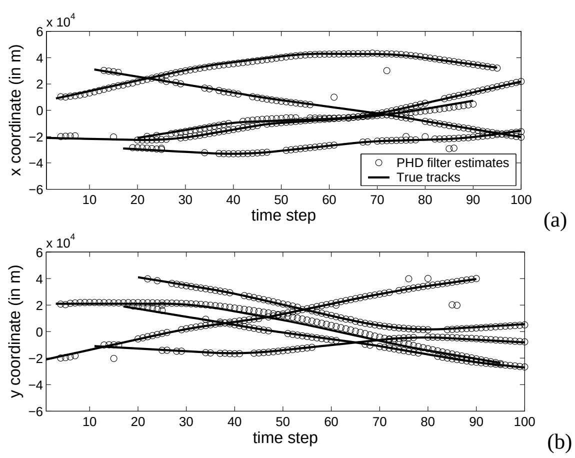

Fig. 2. Position estimates of the Unscented-PHD filter.

Fig. 1 shows the true aircraft trajectories of five aircrafts in the horizontal plane that appear in the surveillance region and disappear at different times and locations. The aircrafts perform a sequence of maneuvers with turn rate in the interval $[-2, 2]^\circ\ s^{-1}$. The position estimates of the PHD filter in Fig. 2 show that the filter successfully tracks the targets in clutter. Occasionally, the filter underestimates the number of aircrafts in the surveillance region and momentarily loses

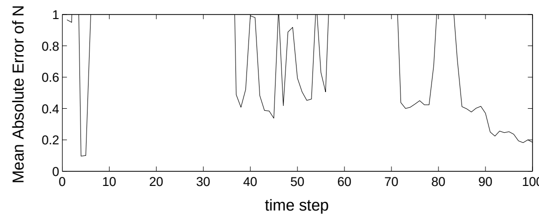

Fig. 3. Mean absolute error of estimated number of targets.

track. Similarly, an overestimate of the number of aircrafts generates false estimates. However, as shown the false estimates do not propagate with time. The mean absolute error in the estimated number of targets averaged over 1×10^3 Monte Carlo runs is shown in Fig. 3. Note that we apply the same pruning procedures and parameters as in [4], [5] for our implementation.

V. CONCLUSIONS

This paper presents a PHD filter for tracking an unknown and time varying number of targets that follow (nonlinear) jump Markov systems models. The proposed algorithm eliminates the need to perform data association gating, track initiation and termination. Simulation results in a nonlinear scenario with an unknown and time-varying number of maneuvering targets observed in clutter shows that the proposed PHD filter has promising performance .

In our approach the multiple models are not "interacting". It is not clear how the PHD filter approach can be extended to interacting multiple models. This is an interesting problem in both theory and practice, which requires further investigation.

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