ON THE BAYES FILTERING EQUATIONS OF FINITE SET STATISTICS

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ABSTRACT

In multi-target tracking, not only the locations of the targets vary with time, the number of targets also varies with time due to targets appearing and disapearing in the scene. The random finite set approach offers a natural and elegant means to model multiple targets and measurements received by the sensors. This framework has culminated in novel multi-target tracking algorithms developed using the tools of Finite Set Statistics (FISST). FISST concepts are not conventional probabilistic concepts and their relationships to conventional probability are not clear. In particular, the validity of the FISST Bayes filter has not been established. This paper presents some connections between FISST and standard probability theory. Moreover, a measure theoretic treatment of the multi-object filtering problem is given to establish the validity of the FISST Bayes filter.

Keywords: Bayesian estimation, Multi-target Tracking, Optimal Filtering, Point Processes, Random Sets.

1. INTRODUCTION

In a multi-target scenario, not only the states of the targets varies with time, the number of targets also varies with time due to new targets being born and old targets dying from time to time [1], [2]. An existing target may or may not generate a measurement at the sensor. Furthermore, the sensor also receives a set of false alarms or clutter, i.e. spurious measurements not originated from any targets. As a result, information on which measurement originates from which target is not available. Thus, multi-target tracking involves jointly estimating the number of targets and their states from noisy sets of measurements with ambiguous origin.

The key to casting multi-target estimation as a Bayesian filtering problem is to conceptually view the target set as a single meta-target and the set of observations collected by the sensor as a single meta-observation. Random finite sets provide natural representations of multi-target states and multi-target measurements. The theory of random sets provides a rigorous unified framework for the seemingly unconnected sub-disciplines of data fusion [5], [6], [7], [9]. The first systematic treatment of multi-sensor multi-target tracking, as part of a unified framework for data fusion using random set theory was Finite Set Statistics (FISST) [5].

In recent years, the random finite set approach for multi-target tracking has been growing in popularity. Novel multi-target tracking algorithms developed from FISST such as the Probability Hypothesis Density filter [8], [9], [10] and its particle implementation [16] have attracted the attention of many reputed target tracking scientists. From a theoretical stand point, central FISST concepts such as set integral and set derivative are not conventional probabilistic concepts. Since measure theoretic probability is the foundation for Bayesian filtering of any kind, it is important to understand its relationship with FISST.

In this paper we take a closer look at the Bayes recursion in FISST first proposed in [5]. The conventional Bayes recursion is based on Bayes rule for probability densities. It is not obvious that Bayes rule for probability densities also applies to set derivatives of belief mass functions, because probability distributions are additive whereas belief mass function are not. To our best knowledge, no rigorous treatment of Bayes rule for set derivatives of belief mass functions are available. Using a result first noted in [16], it is shown that the set derivative of a belief mass function of a RFS is in essence its probability density. This relationship then allows us to rigorously justify the FISST Bayes recursion for multi-target filtering.

2. FINITE SET STATISTICS

This section outlines the key concepts in Finite Set Statistics (FISST). We start by describing how multi-target dynamics and measurements are modeled using random finite sets in Section 2.1. The core concepts such as set integrals, set derivatives and the Bayes recursion in FISST are reviewed

THIS WORK IS SUPPORTED BY AN ARC DISCOVERY GRANT

in Section 2.2. The necessary background on random finite sets are provided in the Appendix.

2.1. Modeling multi-target systems using Random Finite Set

Recall that in single target tracking, the target dynamics is modeled by the Markov transition density $f_{k|k-1}(\cdot|x_{k-1})$, and the measurement process is modeled by the likelihood function $g_k(\cdot|x_k)$. The posterior density, which contains all the information about the target given all the measurements $z_{1:k} = (z_1, ..., z_k)$, can be computed recursively using the Bayes recursion

$$p_{k|k-1}(x_k|z_{1:k-1}) = \int f_{k|k-1}(x_k|x)p_{k-1|k-1}(x|z_{1:k-1})dx \quad (1)$$

$$p_{k|k}(x_k|z_{1:k}) = \frac{g_k(z_k|x_k)p_{k|k-1}(x_k|z_{1:k-1})}{\int g_k(z_k|x)p_{k|k-1}(x|z_{1:k-1})dx}. \quad (2)$$

In a multi-target system, the *multi-target state* X_k is the finite set of all targets at time k. For example, if there are $M(k)$ targets located at $x_{k,1}, \ldots, x_{k,M(k)}$ in $\mathbf{R}^{n_x}$, then $X_k = \{x_{k,1}, \ldots, x_{k,M(k)}\} \subseteq \mathbf{R}^{n_x}$. Thus, the *multi-target state space* is the collection $\mathcal{F}(\mathbf{R}^{n_x})$ of all finite subsets of $\mathbf{R}^{n_x}$. Similarly, the *multi-target measurement* Z_k is the finite set of all measurements at time k. For example if $N(k)$ observations $z_{k,1}, \ldots, z_{k,N(k)}$ in $\mathbf{R}^{n_z}$ are received, then $Z_k = \{z_{k,1}, \ldots, z_{k,N(k)}\} \subseteq \mathbf{R}^{n_z}$. The *multi-target measurement space* is the collection $\mathcal{F}(\mathbf{R}^{n_z})$ of all finite subsets of $\mathbf{R}^{n_z}$.

In a single target system, uncertainty is characterised by modelling the state as a random vector in $\mathbf{R}^{n_x}$ and the measurement as a random vector in $\mathbf{R}^{n_z}$. Likewise, uncertainty in a multi-target system is characterised by modelling the multi-target state as a random finite set in $\mathcal{F}(\mathbf{R}^{n_x})$ and the multi-target measurement as a random finite set in $\mathcal{F}(\mathbf{R}^{n_z})$. A *random finite set* (RFS) is, simply, a finite set-valued random variable just as a random vector is a vector-valued random variable. A more formal discussion of RFS is provided in the Appendix.

To extend the recursion (1)-(2) to multi-target tracking, it is necessary to characterise the multi-target dynamics by a Markov transition density and the multi-target measurement process by a likelihood function. This in turn requires the notion of integration on the spaces $\mathcal{F}(\mathbf{R}^{n_x})$ and $\mathcal{F}(\mathbf{R}^{n_z})$. This reason (among a few others) lead Mahler to develop Finite Set Statistics (FISST) for multi-target tracking [5], [7], [9].

In Finite Set Statistics (FISST), characterisation of multi-target dynamics and measurement are based on belief mass functions rather than probability distributions [5], [7], [9]. The multi-target dynamics can be encapsulated by the parameterised belief mass function

$$\beta_{k|k-1}(S|X_{k-1}) \equiv P(X_k \subseteq S|X_{k-1}),$$

where S is an arbitrary closed subset in $\mathbf{R}^{n_x}$. Likewise, the multi-target measurement process can be encapsulated by the parameterised belief mass function

$$\beta_k(T|X_k) \equiv P(Z_k \subseteq T|X_k),$$

where T is an arbitrary closed subset in $\mathbf{R}^{n_z}$. All information about the multi-target state at time k is captured by the posterior belief mass function given all the multi-target measurements $Z_{1:k} \equiv (Z_1, ..., Z_k)$ up to time k,

$$\beta_{k|k}(S|Z_{1:k}) \equiv P(X_k \subseteq S|Z_{1:k}).$$

However, unlike probability distributions, belief mass functions are non-additive, hence their densities are not defined. FISST provides a non-measure theoretic notion of 'density' for belief mass function via set integrals and set derivatives [5].

2.2. Bayes Filter in Finite Set Statistics

Let $\mathcal{C}(E)$ denote the collection of closed subsets of E (E denotes either $\mathbf{R}^{n_x}$ or $\mathbf{R}^{n_z}$). Let λ_K denote the volume measure on E in units of K. Note that $\lambda_K = K\lambda$, where λ denotes the unitless Lebesque measure. The set derivative of a function $F : \mathcal{C}(E) \to [0, \infty)$ at a point $x \in E$ is a mapping $(d_K F)_x : \mathcal{C}(E) \to [0, \infty)$ defined as[1]

$$(d_K F)_x(S) \equiv \lim_{\lambda_K(\Delta_x)\to 0} \frac{F(S \cup \Delta_x) - F(S)}{\lambda_K(\Delta_x)}, \quad (3)$$

where $\lambda_K(\Delta_x)$ is the volume of a neighbourhood Δ_x of x in units of K. This is a simplified version of the complete definition given in [5]. Furthermore, the set derivative at a finite set $X = \{x_1, ..., x_n\}$ is defined by the recursion

$$(d_K F)_{\{x_1,...,x_n\}}(T) \equiv (d_K(d_K F)_{\{x_1,...,x_{n-1}\}})_{x_n}(T),$$

where $(d_K F)_\emptyset \equiv F$ by convention[2]. Note that $(d_K F)_X(S)$ has unit of $K^{-|X|}$, where $|X|$ denote the cardinality or number of elements of X. Hence, for a fixed $S \subseteq E$ the set derivatives $(d_K F)_X(S)$ and $(d_K F)_Y(S)$ have different units if $|X| \neq |Y|$. In the case where volume in E is unitless, λ_K in definition (3) becomes λ, we use the notation $(dF)_X$ to denote a unitless set derivative.

[1] In [5] the notation $\frac{\delta F}{\delta x}(S)$ was used for the set derivative $(d_K F)_x(S)$. Note our use of the subscript K to emphasize the dependence on the unit of measurement.

[2] In [5] the notation $\frac{\delta F}{\delta X}(S)$ was used for the set derivative $(d_K F)_X(S)$.

Let f be a function defined by $f(X) = (d_K F)_X(\emptyset)$. Then the set integral of f over a (closed) subset $S \subseteq E$ is defined as [5], [7], [9], [10][3]

$$\int_S f(X)\delta_K X \equiv \sum_{i=0}^{\infty} \frac{1}{i!} \int_{S^i} f(\{x_1, ..., x_i\})\lambda_K^i(dx_1...dx_i). \tag{4}$$

In the unitless case, i.e. λ_K in definition (4) becomes λ, we use the notation $\int_S f(X)\delta X$ to denote a unitless set integral.

The set integral and set derivative are related by the following generalised fundamental theorem of calculus,

$$f(X) = (d_K F)_X(\emptyset) \text{ if and only if } F(S) = \int_S f(X)\delta_K X.$$

Given the notion of integral and derivative, we can define the *FISST multi-target posterior density* $\pi_{k|k}(X_k|Z_{1:k})$ as the set derivative of $\beta_{k|k}(\cdot|Z_{1:k})$ at X_k; the *FISST multi-target transition density* $\varphi_{k|k-1}(X_k|X_{k-1})$ as the set derivative of $\beta_{k|k-1}(\cdot|X_{k-1})$ at X_k; and the *FISST multi-target likelihood* $\rho_k(Z_k|X_k)$ as the set derivative of $\beta_k(\cdot|X_k)$ at Z_k. The prefix FISST is used to distinguish these densities from conventional densities. The interested reader is referred to [5], [7], [9], [10] for details on how FISST is used to calculate $\varphi_{k|k-1}$ and ρ_k from individual models of targets and sensors. The FISST multi-target transition density $\varphi_{k|k-1}$ incorporates all aspects of motion of multiple targets such as the time-varying number of targets, individual target motion, target birth, spawning and target interactions. The FISST multi-target likelihood ρ_k incorporates all sensor behaviour such as measurement noise, sensor field of view (i.e. state-dependent probability of detection) and clutter models.

In [5] pp.238-239, it was argued that the Bayes recursion (1)-(2) generalises to the multi-target case in an obvious fashion i.e.

$$\begin{aligned} &\pi_{k|k-1}(X_k|Z_{1:k-1}) \\ &= \int \varphi_{k|k-1}(X_k|X)\pi_{k-1|k-1}(X|Z_{1:k-1})\delta_K X \quad (5) \\ &\pi_{k|k}(X_k|Z_{1:k}) \\ &= \frac{\rho_k(Z_k|X_k)\pi_{k|k-1}(X_k|Z_{1:k-1})}{\int \rho_k(Z_k|X)\pi_{k|k-1}(X|Z_{1:k-1})\delta_K X}. \quad (6) \end{aligned}$$

At this stage the mathematically minded reader might question the validity of the recursion (5)-(6). Even though it is well-known that the Bayes recursion (1)-(2) generalises to more general spaces with sufficiently 'nice' structures and consistent notions of integration, the Bayes recursion was only established for probability densities. It is not obvious that Bayes rule for probability densities would hold for set derivatives of belief mass function, because probability distributions are additive whereas belief mass function are not. A rigorous treatment of Bayes rule for probability densities requires deep results in conditional probability [13] pp. 230-231. To our best knowledge no such rigorous treatment of Bayes rule for set derivatives of belief mass functions is available. In the next section, we will address this issue and establish that (5)-(6) is valid through standard measure theoretic arguments.

[3] In [10], [5] pp. 141-142, the set integral is defined for any real or vector valued function f. This is probably a typographical error as it does not make sense due to unit incompatability. Note also our use of the subscript K to emphasize the dependence on the unit of measurement.

3. BAYES FILTER

In this section, we characterise the multi-target dynamics and measurement process by probability distributions instead of belief mass functions (as done in Section 2). Based a on result first noted in [16], it is shown in section 3.1 that the set derivative of a belief mass function of a RFS is closely related to its probability density. Using this relationship together with a conventional measure theoretic formulation of the multi-target tracking problem allow us to establish, in section 3.2, that the FISST Bayes recursion (5)-(6) is indeed correct.

3.1. Relationship between set derivative and density

Let μ be an unnormalised distribution of a Poisson point process (see Apendix A) with a uniform rate of K^{-1} [4], [12] i.e.

$$\mu(\mathcal{T}) = \sum_{i=0}^{\infty} \frac{\lambda^i(\chi^{-1}(\mathcal{T}) \cap E^i)}{i!}. \tag{7}$$

where $\mathcal{T}$ is an arbitrary Borel subset of $\mathcal{F}(E)$, λ^i is the ith product (unitless) Lebesque measure, $\chi : \uplus_{i=0}^{\infty} E^i \to \mathcal{F}(E)$ is a mapping of vectors to sets defined by $\chi([x_1, ..., x_i]^T) = \{x_1, ..., x_i\}$, and $\uplus$ denotes disjoint union. The mapping χ is measurable [15] and μ is well defined. Note that for simplicity the mapping χ is often omitted from the above equation. Finite sets can also be written as vectors and vice-versa, i.e. $\mathcal{T}$ and $\chi^{-1}(\mathcal{T})$ can be used interchangeably in the integral or measure of interest, and the meaning can be interpreted from the context of the expression.

The integral of a real valued function $f : \mathcal{F}(E) \to [0, \infty)$ with respect to the measure (7) is given by [4], [12] (see Appendix)

$$\begin{aligned} &\int_{\mathcal{T}} f(X)\mu(dX) \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\chi^{-1}(\mathcal{T})\cap E^i} f(\{x_1, ..., x_i\})\lambda^i(dx_1...dx_i) \end{aligned} \tag{8}$$

With the notion of integration on the space $\mathcal{F}(E)$ in place, probability density on $\mathcal{F}(E)$ can now be defined. For a RFS Ξ with probability distribution P_Ξ, the *probability density* $\frac{dP_\Xi}{d\mu}$ with respect to the measure μ is a function $p_\Xi : \mathcal{F}(E) \to [0, \infty)$ that satisfies

$$P_\Xi(\mathcal{T}) = \int_{\mathcal{T}} p_\Xi(X)\mu(dX),$$

for any Borel subset $\mathcal{T} \subseteq \mathcal{F}(E)$.

In [16], it was noted that for any arbitrary closed set S in E,

$$\int_{\chi(\uplus_{i=0}^{\infty} S^i)} f(X)\mu(dX) = \int_S f(X)\delta X.^4 \tag{9}$$

This implies that the unitless set derivative of a belief mass function and the probability density w.r.t. μ are the same. This extends trivially to set derivative with unit.

Proposition 1. *Suppose that Ξ is a RFS on E with probability distribution P_Ξ and belief mass function β_Ξ. If P_Ξ is absolutely continuous with respect to the measure μ defined by (7), then*

$$\frac{dP_\Xi}{d\mu}(X) = K^{|X|}(d\beta_\Xi)_X(\emptyset). \tag{10}$$

Proof: Multiplying and dividing by appropriate units for the function f and the Lebesque measure in (9) yields (11), i.e.

$$\begin{aligned}
&\int_{\chi(\uplus_{i=0}^{\infty} S^i)} f(X)\mu(dX) \\
=\ & \sum_{i=0}^{\infty} \frac{1}{i!} \int_{S^i} f(\{x_1, ..., x_i\})\lambda^i(dx_1...dx_i) \\
=\ & \sum_{i=0}^{\infty} \frac{1}{i!} \int_{S^i} K^{-i} f(\{x_1, ..., x_i\})\lambda_K^i(dx_1...dx_i) \\
=\ & \int_S K^{-|X|} f(X)\delta_K X. \qquad (11)
\end{aligned}$$

Let $p_\Xi = dP_\Xi/d\mu$. Then, it follows from Eq. (11) that, for any closed $S \subseteq E$,

$$\begin{aligned}
\int_S K^{-|X|} p_\Xi(X)\delta X &= \int_{\chi(\uplus_{i=0}^{\infty} S^i)} p_\Xi(X)\mu(dX) \\
&= P_\Xi(\chi(\uplus_{i=0}^{\infty} S^i)) = \beta_\Xi(S).
\end{aligned}$$

Since S is arbitrary, from the FISST fundamental theorem of calculus $K^{-|X|} p_\Xi(X) = (d\beta_\Xi)_X(\emptyset)$. QED.

Recall that $(d\beta_\Xi)_X(\emptyset)$ has unit of $K^{-|X|}$ and hence $K^{|X|}(d\beta_\Xi)_X(\emptyset)$ is unitless. Proposition 1 implies that the set derivative of the belief mass function β_Ξ, when striped off its unit, is the probability density p_Ξ with respect to the dominating measure μ. In other words, the unitless set derivative of the belief mass function of a RFS is its probability density. It is important to point out that while the probability density is unitless, it is still dependent on the unit of measurement K. This is because the dominating measure μ is an unnormalised distribution of a Poisson point process with a uniform rate of K^{-1}. Changing the unit of measurement changes the dominating measure, which in turn affect the resulting probability density.

[4] The practice of omitting χ for simplicity was followed in [16], we explicitly use χ in this paper to avoid ambiguity.

3.2. Measure theoretic formulation

For any Borel subsets $\mathcal{U} \subseteq \mathcal{F}(\mathbf{R}^{n_x})$, $\mathcal{V} \subseteq \mathcal{F}(\mathbf{R}^{n_z})$ let

$$P_{k|k}(\mathcal{U}|Z_{0:k}) \equiv P(X_k \in \mathcal{U}|Z_{1:k})$$

denote the (posterior) probability of the multi-target state given all the observations $Z_{1:k} = (Z_1, ..., Z_k)$ up to time k, let

$$P_{k|k-1}(\mathcal{U}|X_{k-1}) \equiv P(X_k \in \mathcal{U}|X_{k-1})$$

denote the Markov transition probability and let

$$P_k(\mathcal{V}|X_k) \equiv P(Z_k \in \mathcal{V}|X_k)$$

denote the probability of the measurement given a multi-target state X_k.

Let μ_s and μ_o be dominating measures of the form (7) on the Borel subsets of $\mathcal{F}(\mathbf{R}^{n_x})$ and $\mathcal{F}(\mathbf{R}^{n_z})$ respectively. Then, the *multi-target posterior density* $p_{k|k}(\cdot|Z_{1:k})$, the *multi-target transition density* $f_{k|k-1}(\cdot|X_{k-1})$ and the *multi-target likelihood* $g_k(\cdot|X_k)$ are the densities of $P_{k|k}(\cdot|Z_{1:k})$ w.r.t. μ_s, $P_{k|k-1}(\cdot|X_{k-1})$ w.r.t. μ_s, and $P_k(\cdot|X_k)$ w.r.t. μ_o respectively i.e.

$$\begin{aligned}
P_{k|k}(\mathcal{U}|Z_{1:k}) &= \int_{\mathcal{U}} p_{k|k}(X_k|Z_{1:k})\mu_s(dX_k), \\
P_{k|k-1}(\mathcal{U}|X_{k-1}) &= \int_{\mathcal{U}} f_{k|k-1}(X_k|X_{k-1})\mu_s(dX_k), \\
P_k(\mathcal{V}|X_k) &= \int_{\mathcal{V}} g_k(Z_k|X_k)\mu_o(dZ_k).
\end{aligned}$$

The statistical properties of the target dynamics are captured by the multi-target transition density $f_{k|k-1}$ in an analogous fashion to the single-target transition density. Likewise, the statistical properties of the measurement process can now be described by the multi-target likelihood g_k in an analogous fashion to the single-target likelihood function. Like its FISST counter part, the multi-target transition density $f_{k|k-1}$ incorporates all aspects of motion of multiple targets such as the time-varying number of targets, individual target motion, target birth, spawning and target interactions. Similarly, the multi-target likelihood g_k incorporates all sensor behaviour such as measurement noise, sensor field of view (i.e. state-dependent probability of detection) and clutter models.

Since the Bayes recursion (12)-(2) generalises for probability densities on more general spaces with sufficiently 'nice' structures and consistent notions of integration, the optimal multi-target Bayes filter is given by the recursion

$$p_{k|k-1}(X_k|Z_{1:k-1}) = \int f_{k|k-1}(X_k|X) p_{k-1|k-1}(X|Z_{1:k-1}) \mu_s(dX) \quad (12)$$

$$p_{k|k}(X_k|Z_{1:k}) = \frac{g_k(Z_k|X_k) p_{k|k-1}(X_k|Z_{1:k-1})}{\int g_k(Z_k|X) p_{k|k-1}(X|Z_{1:k-1}) \mu_s(dX)}. \quad (13)$$

Let K_s and K_o denote the units of volume in the spaces $\mathbf{R}^{n_x}$ and $\mathbf{R}^{n_z}$ respectively. Using Proposition 1, we have

$$\begin{aligned} p_{k-1|k-1}(X|Z_{1:k-1}) &= K_s^{|X|} \pi_{k-1|k-1}(X|Z_{1:k-1}), \\ p_{k|k-1}(X|Z_{1:k-1}) &= K_s^{|X|} \pi_{k|k-1}(X|Z_{1:k-1}), \\ f_{k|k-1}(X|X_{k-1}) &= K_s^{|X|} \varphi_{k|k-1}(X|X_{k-1}), \\ p_{k|k}(X|Z_{1:k}) &= K_s^{|X|} \pi_{k|k}(X|Z_{1:k}), \\ g_k(Z|X_k) &= K_o^{|Z|} \rho_k(Z|X_k). \end{aligned}$$

Substituting these into (12)-(13) and using (11) yields the recursion (5)-(6). Thus the FISST Bayes recursion (5)-(6) can be derived from the conventional Bayes recursion.

4. CONCLUSIONS

In this paper, it has been established that the set derivative for belief mass function is in essence a probability density. It has also been shown that the FISST Bayes filter can be derived using classical probability tools. These results illustrate the connection between FISST and conventional probability theory. More importantly, they highlight the role of FISST in conventional treatment of random finite sets, where it is not clear how probability densities are calculated. In a multi-target tracking context, where models of individual targets and sensors are given in terms of probability distributions on the single-object state space, FISST provides the tools for calculating the multi-target transition density and likelihood required for the Bayes multi-target filter.

5. APPENDIX: RANDOM FINITE SETS

For completeness, this Appendix outlines the basics of random finite sets (RFS) or simple finite point processes[5]. Background material on RFS are abundant in the point processes literature; see for example [3], [14]. However, works with an inclination to multi-target tracking are quite new; the major body of work appears to be that of Mahler [5], [7], [9]. The monograph [7] and [9] are excellent introductions accessible to a wide range of readers.

Given a locally compact Hausdorff separable space E, let $\mathcal{F}(E)$ denote the collection of finite subsets of E. The topology on $\mathcal{F}(E)$ is taken to be the myopic or Mathéron topology [11]. A *random finite set* Ξ on E is defined as a measurable mapping

$$\Xi : \Omega \to \mathcal{F}(E),$$

where Ω is a sample space with a probability measure P defined on $\sigma(\Omega)$. The probability measure P induces a *probability law* for Ξ, which can be specified in terms of a probability distribution, a void probability or a belief function. The most natural description of the probability law for Ξ is the probability distribution P_Ξ defined for any Borel subset $\mathcal{T}$ of $\mathcal{F}(E)$ by

$$P_\Xi(\mathcal{T}) = P(\Xi^{-1}(\mathcal{T})) = P(\{\omega : \Xi(\omega) \in \mathcal{T}\}).$$

However, from random set theory [5], [11], the probability law for Ξ can also be given in terms of the *belief mass function* β_Ξ defined for any closed subset S of E by

$$\beta_\Xi(S) = P(\{\omega : \Xi(\omega) \subseteq S\}).$$

A third equivalent description, closely related to the belief mass function, is the *void probability* ς_Ξ [3], [14], [15], which is defined for any closed subset S of E by

$$\varsigma_\Xi(S) = P(\{\omega : |\Xi(\omega) \cap S| = 0\}) = \beta_\Xi(S^c),$$

where $|X|$ denotes the number of elements in X.

The simplest class of RFSs are the *Poisson point processes*. A Poisson point process Υ is a RFS characterised by the property that for any k disjoint Borel subsets $S_1, ..., S_k$ of E, the random variables $|\Upsilon \cap S_1|, ..., |\Upsilon \cap S_k|$ are independent and Poisson. Let $v_\Upsilon(S)$ denote the mean of the Poisson random variable $|\Upsilon \cap S|$. Then v_Υ defines a (unitless) measure on the Borel subsets of E, and is called the *intensity measure* of Υ [14], [15]. The probability distribution of Υ is given by [4], [12]

$$P_\Upsilon(\mathcal{T}) = e^{-v_\Upsilon(E)} \sum_{i=0}^{\infty} \frac{v_\Upsilon^i(\chi^{-1}(\mathcal{T}) \cap E^i)}{i!}, \quad (14)$$

where v_Υ^i denotes the ith product measure of v_Υ and $\chi : \uplus_{i=0}^{\infty} E^i \to \mathcal{F}(E)$ is the mapping of vectors to finite sets defined for each i by $\chi([x_1, ..., x_i]^T) = \{x_1, ..., x_i\}$. The mapping χ is measurable [15] and hence P_Υ is well defined. A word of caution, it is common practice in the stochastic

[5]A simple finite point process set does not allow repeated elements and only contains a finite number of elements.

geometry literature to write the measure (14) with the following abuse of notation [4], [12]

$$P_\Upsilon(\mathcal{T}) = e^{-v_\Upsilon(E)} \sum_{i=0}^{\infty} \frac{v^i_\Upsilon(\mathcal{T} \cap E^i)}{i!},$$

where it is implicit that $\mathcal{T} := \chi^{-1}(\mathcal{T})$, i.e. vectors are considered as finite sets and vice-versa depending on the context of the expression.

The integral of a measurable function $f : \mathcal{F}(E) \to \mathbf{R}$ with respect to the measure

$$\mu(\mathcal{T}) = \sum_{i=0}^{\infty} \frac{v^i_\Upsilon(\chi^{-1}(\mathcal{T}) \cap E^i)}{i!},$$

is given by [4], [12]

$$\begin{aligned} & \int_{\mathcal{T}} f(X)\mu(dX) \\ = & \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\chi^{-1}(\mathcal{T})\cap E^i} f(\{x_1, ..., x_i\}) v^i_\Upsilon(dx_1...dx_i). \end{aligned}$$

This is straight forward to verify using the countable additivity of measure. Decompose $\mathcal{T} = \uplus_{i=0}^{\infty} \mathcal{T}_i$, where $\mathcal{T}_i$ is the subset of $\mathcal{T}$ which contains all (finite) subsets with i elements, and note that $\chi^{-1}(\mathcal{T}_i) \cap E^i = \chi^{-1}(\mathcal{T}) \cap E^i$, then

$$\begin{aligned} & \int_{\mathcal{T}} f(X)\mu(dX) \\ = & \sum_{i=0}^{\infty} \int_{\mathcal{T}_i} f(X)\mu(dX) \\ = & \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\chi^{-1}(\mathcal{T}_i)\cap E^i} f(\{x_1, ..., x_i\}) v^i_\Upsilon(dx_1...dx_i) \\ = & \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\chi^{-1}(\mathcal{T})\cap E^i} f(\{x_1, ..., x_i\}) v^i_\Upsilon(dx_1...dx_i) \end{aligned}$$

For any Borel subset S of E, let $\lambda_K(S)$ be the Lebesque measure (or volume) of S in units of K. The density of v_Υ w.r.t. λ_K (if one exists) is called an *intensity function* or *rate* of Υ and has units of K^{-1}. A Poisson point process is completely characterised by its intensity measure (or equivalently its rate). A Poisson point process with a uniform rate of K^{-1} has intensity measure $\lambda \equiv \lambda_K / K$ and its probability distribution is given by [4], [12] i.e.

$$P_\Upsilon(\mathcal{T}) = e^{-\lambda(E)} \sum_{i=0}^{\infty} \frac{\lambda^i(\chi_i^{-1}(\mathcal{T}) \cap E^i)}{i!}. \qquad (15)$$

This implicitly assumes that is $\lambda(E)$ finite.

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