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A STOCHASTIC GEOMETRIC APPROACH TO SENSOR ARRAY PROCESSING

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ABSTRACT

A new unified mathematical framework for sensor array processing is proposed. The proposed framework combines Bayesian estimation with stochastic geometry to accommodate prior information, uncertainty in array parameters, and unknown and stochastically time-varying number of nonstationary sources. A system model for a common signal setting is constructed to demonstrate the proposed framework.

Index Terms— sensor array processing, Bayesian estimation, random sets

1. INTRODUCTION

In sensor array processing, the primary goal is to estimate signals and/or parameters by fusing spatial and temporal information captured by sampling a wave field (in space-time) with an array of sensors. Array processing is a core problem in signal processing with a wealth of literature, see for example the books [1–3], and lies at the heart of a wide range of disciplines from radio astronomy, radar, sonar to wireless communication and medical imaging. Current approaches are often constrained by simplifying assumptions such as stationarity/cyclostationarity of the signals, known sensor positions, availability of complete gain/phase/mutual coupling calibration data, etc. [3]. These assumptions do not always hold in practice and if modelling errors were not taken into account, severe performance degradation can arise. In particular, a popular fix for the nonstationarity of the signal is to assume that the signal statistics slowly vary in such a way that the signal can be treated as stationary/cyclostationary within the data window. This approach does not formally exploit prior information or knowledge on the spatiotemporal characteristics of the signals, and may break down in real-world scenarios when signal sources appear and disappear randomly in time as well as moving in space. Uncertainty in the array parameters is often assumed to be bounded (with known bounds) and robustness is achieved by hedging against worse case scenarios. Consequently, the solutions are far too conservative and, in general, the resulting optimization problems are non-convex with no reliable solution methods.

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This paper proposes a new unified mathematical framework for sensor array processing by combining Bayesian estimation [4] with stochastic geometry [5–7] in order to accommodate prior information, uncertainty in array parameters, and unknown and stochastically time-varying number of nonstationary sources. Bayesian approaches have previously been applied to various aspects of array processing albeit in isolation, such as array calibration [8], subspace estimation [9], beamforming [10], array geometry [11], robust array processing [12], and direction-of-arrival estimation [13, 14]. To the best of the authors' knowledge, a unified framework combining Bayesian estimation/inferencing and stochastic geometric modelling, from a fundamental systems theoretic perspective for sensor array processing has not been attempted.

Our philosophy departs from the traditional formulations that are based on optimizing the signal-to-interference-plus-noise ratio (SINR) or array beam pattern over a data window where signals are assumed to be stationary/cyclostationary. Optimal solutions in such formulations do not necessarily give better estimates than directly estimating the set of underlying signals/parameters themselves. Our framework adopts the direct approach that explicitly seeks optimal estimates of the set of underlying signals/parameters, whose number and values vary stochastically with time. The salient features of this approach are

- Treating the system of sensors and sources as a multi-object system, thereby accommodating a time-varying number of nonstationary signal sources (and even sensors), and allowing for a mathematically consistent notion of error metrics since distance for sets are well-defined,
- The combination of the Bayesian paradigm with stochastic geometric modelling to provide a top-down unifying framework for array processing,
- The ability to accommodate uncertainty in array parameters and array calibration, and being amenable for on-line estimation as well as for integrating array processing with source tracking.

2. RANDOM FINITE SET FRAMEWORK FOR BAYESIAN SENSOR ARRAY PROCESSING

The proposed framework integrates the relevant concepts from stochastic geometry, filtering theory, and sensor array processing, in a systematic and coherent manner that enables system modelling, Bayesian estimation, and robust solutions for array processing.

2.1. System Modeling

The premise is to formalize a state space description of the system of sensor array and signal sources. The system model is specified in terms of dynamic and measurement models for the signal sources and array. Prior information about the system such as birth/death, motion, sensor noise and other application specific parameters for example transmit waveform (in radar), or modulation scheme (in communications), are captured in the initial prior, transition kernel and likelihood function.

We consider an array as a collection of individual sensors arranged in a certain geometric configuration, which in practice, have some associated uncertainty. Each sensor and signal source has its own possibly latent state which could be static or changing with time. For example the state vector of a sensor could be a combination of its spatial locations, noise parameters, gain/phase information, and the state of a transmitter could be the spatial location of the source (or simply its DOA), the modulation scheme and modulation parameters, and signal strength.

The collection of multiple sensors and signal sources together can be considered as a multi-object system. The number of sources can vary randomly in time because signal sources can appear or disappear. The number of sensors are fixed, and each sensor has its own identity. At time k there are two sets of signal sources, the set of signals of interest (SoIs) X_k and the set of interference signals I_k. Exact knowledge of the number, and values of the SoIs and interference signals are not available. The main goal is to estimate the SoIs from the array measurements.

To model uncertainty in the finite sets X_k and I_k, we invoke the notion of multi-object probability density or probability density for random finite sets from stochastic geometry [4–7].

To cast the array processing problem in a Bayesian framework we employ probabilistic models for the time evolution of the collection of nonstationary signal sources (including interference sources). It is assumed that X_k and I_k evolve in time jointly according to a Markov chain with transition kernel $f(X_k, I_k|X_{k-1}, I_{k-1})$ initial prior $p(X_0, I_0)$ which encapsulate all aspects of the birth, death, motion and prior information for individual sources. We adopt the standard assumption that the SoIs and interferences are statistically independent $p(X_0, I_0) = p_X(X_0)p_I(I_0)$. The signals are observed by an array of N sensors each with state vector r_i, $i = 1, \dots, N$. The array measurement $\mathbf{z}_k = [z_{1,k}, \dots, z_{N,k}]^T$ is modelled by the measurement likelihood function $g(\mathbf{z}_k|X_k, I_k, \mathbf{r})$, where $\mathbf{r} = [r_1, \dots, r_N]$. Specific to individual applications, different likelihood functions and transition kernels encapsulate all aspects of the system model.

2.2. Bayesian Estimation

The essence of the Bayesian approach is that conceptually, all information about the system is captured in the joint posterior density of the SoIs and interferences, which can be computed recursively via the Bayes recursion:

$$p(X_{0:k}, I_{0:k}|\mathbf{z}_{1:k}, \mathbf{r}) \propto g(\mathbf{z}_k|X_k, I_k, \mathbf{r}) f(X_k, I_k|X_{k-1}, I_{k-1}) \times p(X_{0:k-1}, I_{0:k-1}|\mathbf{z}_{1:k-1}, \mathbf{r}).$$

where $X_{0:k} \triangleq [X_0, ..., X_k]$, $I_{0:k} \triangleq [I_0, ..., I_k]$ and $\mathbf{z}_{1:k} \triangleq [\mathbf{z}_1, ..., \mathbf{z}_k]$. The posterior density of the SoIs can be obtained by marginalizing over the interference signal set:

$$p(X_{0:k}|\mathbf{z}_{1:k}) = \int p(X_{0:k}, I_{0:k}|\mathbf{z}_{1:k}) \delta I_{0:k}$$

where the integral above is a multiple set integral since $I_{0:k}$ is an array of finite sets. The ultimate goal is to compute the posterior distribution of the system (or SoIs), from which we can infer the "best" estimate of the set of SoIs. For real-time applications, we are interested in the filtering density which captures the system state at the current time and can be computed recursively by a prediction and update step:

$$p(X_k, I_k|\mathbf{z}_{1:k-1}, \mathbf{r}) = \int f(X_k, I_k|X_{k-1}, I_{k-1}) \times p(X_{k-1}, I_{k-1}|\mathbf{z}_{1:k-1}, \mathbf{r}) \delta X_{k-1} \delta I_{k-1}$$

$$p(X_k, I_k|\mathbf{z}_{1:k}, \mathbf{r}) \propto g(\mathbf{z}_k|X_k, I_k, \mathbf{r}) p(X_k, I_k|\mathbf{z}_{1:k-1}, \mathbf{r}).$$

Bayes optimal estimators are then required to calculate the final outputs. Since the number of sources is generally not known a priori, then standard estimators are no longer applicable since the underlying state variable is a set of states. In this case exact Marginal Multi-object (MaM) or Joint Multi-object (JoM) estimators [4], or approximate Probability Hypothesis Density (PHD) or Cardinalized PHD estimator [4] estimators can be considered. Propagation of the full posterior is generally computationally intensive although not intractable. Approximations may be more viable, such as those based on moments and cardinality distributions in the spirit of the PHD and CPHD filters [4, 15–17]. Another alternative is based on parameterized approximations with intuitive interpretations such as the Multi-Bernoulli filters [18, 19].

2.3. Robust Solutions

In robust array processing the aim is to estimate the set of SoIs subject to uncertainty in parameters such as array element position and response, array geometry, and other array characteristics. Given the posterior density of the array system state and a prior p on array parameter $\mathbf{r}$, it is possible to hedge against uncertainty in $\mathbf{r}$ by using the total probability theorem to marginalize the array parameter to obtain the unconditional posterior or unconditional filtering density:

$$\begin{aligned} p(X_{0:k}, I_{0:k}|\mathbf{z}_{1:k}) &= \int p(X_{0:k}, I_{0:k}|\mathbf{z}_{1:k}, \mathbf{r})p(\mathbf{r})d\mathbf{r} \\ p(X_k, I_k|\mathbf{z}_{1:k}) &= \int p(X_k, I_k|\mathbf{z}_{1:k}, \mathbf{r})p(\mathbf{r})d\mathbf{r}. \end{aligned}$$

Computationally this problem is much more challenging than the case with known array parameters. A related problem is array calibration. Here the array parameter $\mathbf{r}$ is uncertain, and the objective is to estimate it rather than the set of SoIs. This problem can be formulated as a data fitting problem, i.e. choose the $\mathbf{r}$ that "best" explains the array measurement, in the sense of maximum likelihood or Bayes optimality. Conceptually, the posterior distribution of the array parameters is given by

$$p(\mathbf{r}|\mathbf{z}_{1:k}) = \int\int p(X_{0:k}, I_{0:k}|\mathbf{z}_{1:k}, \mathbf{r})p(\mathbf{r})\delta X_{0:k}\delta I_{0:k}.$$

The solution to the array calibration problem is the $\mathbf{r}$ that maximizes the above posterior, or the expected value of $\mathbf{r}$ conditional on the array measurement. The latter solution also minimizes the mean squared error. The multiple set integrals are numerically intractable, but it is possible to use Monte Carlo approximation. However, due to the dimensionality of the integral, such approximation can be computationally expensive. The problem becomes much more tractable if training data is used. In this case the signal sources (both SoIs and interferences) are known a priori and we can estimate $\mathbf{r}$ based on the probability density of the array parameter conditional on the array measurements and training data $p(\mathbf{r}|\mathbf{z}_{1:k}, X_{0:k}, I_{0:k})$, which can be obtained via a straighforward application of Bayes rule, or alternatively we can estimate $\mathbf{r}$ based on $p(X_{0:k}, I_{0:k}, \mathbf{r}|\mathbf{z}_{1:k})$, the joint probability density of the signal sources and the array parameter, for fixed array measurements and training signals.

3. EXAMPLE FORMULATION

We assume a setting similar to the standard array processing formulation in which there are multiple signal sources with no interference, in addition to a fixed and known a priori array geometry and array parameters. In contrast to standard formulations however we allow for an unknown and time varying number of multiple signal sources. We further allow any array geometry and hence are not restricted to the standard linear or circular arrays. The resultant formulation essentially comprises the derivation of the measurement likelihood and transition kernel. With the system model, it is then possible to apply Bayesian estimation techniques, including generalizations to robust solutions, in order to estimate the underlying SoIs from the array measurements. We focus on the system modelling here and defer estimation to an expanded study.

3.1. Measurement Likelihood

Assume at time k that a total of $M(k)$ number of signal sources are impinging on the N element sensor array. The DOAs of the signals at time k are denoted by $\phi_{1,k}, \ldots, \phi_{M(k),k}$. Without loss of generality assume for convenience that the multi-object state is simply the set of DOAs $X_k = \{\phi_{1,k}, \ldots, \phi_{M(k),k}\}$. The sources have fixed but unknown amplitudes $\alpha_{1,k}$, ..., $\alpha_{M(k),k}$ and steering vectors $\mathbf{s}(\phi_{1,k})$,...,$\mathbf{s}(\phi_{M(k),k})$. Each individual source resides in the far-field and emits narrow-band waves. If the output of each array element at time k is denoted $z_{1,k}, ..., z_{N,k}$ then output of the sensor array $\mathbf{z}_k = [z_{1,k}, ..., z_{N,k}]^T$ is given by

$$\mathbf{z}_k = \sum_{m=1}^{M(k)} \alpha_m \mathbf{s}(\phi_{m,k}) + \mathbf{n}_k$$

where $\mathbf{n}_k$ is a zero mean white Gaussian array measurement noise vector with diagonal covariance matrix $\sigma_n^2 \mathbf{I}_N$. Now collect the source amplitudes into an amplitude vector $\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_M]^T$ and the steering vectors into a matrix $\mathbf{S}(X_k) = [\mathbf{s}(\phi_1) \ \cdots \mathbf{s}(\phi_{M(k),k})]$. The likelihood of the array measurement vector $\mathbf{z}_k$ given the DOAs X_k is

$$g(\mathbf{z}_k|X_k) = \mathcal{N}(\mathbf{z}_k, \mathbf{S}(X_k)\boldsymbol{\alpha}, \sigma_n^2 \mathbf{I}_N)$$

Since the amplitudes are assumed unknown, a straightforward approach to evaluation of the likelihood is to substitute for the maximum likelihood amplitude estimates obtained by solving $\frac{\partial}{\partial \boldsymbol{\alpha}} g(\mathbf{y}|\Phi, \boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}} = 0$ giving $\hat{\boldsymbol{\alpha}} = [\mathbf{S}(X_k)\mathbf{S}(X_k)^H]^{-1}\mathbf{S}(X_k)^H \mathbf{y}_k$. A more accurate but computationally expensive approach is to augment the unknown amplitude into the state of each individual source thereby enabling joint estimation of the source DOAs and amplitude.

3.2. Transition Kernel

The dynamic changes of the signal sources, including the birth (appearance of a new one), death (disappearance of an existing) and transition (physical movement), are modelled by a Markov transition kernel. Recall that the DOAs of the signals at time $k-1$ are denoted by $\phi_{1,k-1}, \ldots, \phi_{M(k-1),k-1}$ and without loss of generality that the multi-object state is simply the set X_{k-1} of DOAs. The dynamic model and hence transition kernel is built upon the following assumptions.

Each source with DOA ϕ at time $k-1$ survives or persists to time k with state dependent survival probability $p_{S,k}(\phi)$ and consequently ceases to exist with death probability $1 - p_{S,k}(\phi)$. Conditional upon survival or persistence from time $k-1$ to time k, each source moves or transitions from an old DOA ϕ_{k-1} at $k-1$ to a new DOA ϕ_k at time k with probability density $\xi(\phi_k|\phi_{k-1})$. The transition density for individual sources $\xi(\cdot|\cdot)$ thus completely describes the physical motion of each source. A simple motion model is a random walk on the DOA or a Gaussian transition density. For more complicated motions such as constant velocity model on a 3-D position and velocity the transition density for source DOAs needs to be approximated. New sources appear according to a Poisson birth model, with mean rate $\lambda_{B,k}$, independently with initial DOA ϕ with probability density $p_{B,k}(\phi)$.

Under the above assumptions, it can be shown that the transition kernel, or probability density X_{k-1} transitions to X_k (of DOAs) from time $k-1$ to k is given by [4]

$$f(X_k|X_{k-1}) = \sum_{W\subseteq X_k} s(W|X_{k-1})b(X_k - W)$$
$$s(W|X_{k-1}) = [1-p_{S,k}(\cdot)]^{X_{k-1}} \times \sum_{\tau\in\mathcal{T}(W,X_{k-1})}\left[\frac{p_{S,k}(\tau(\cdot))\xi_{k|k-1}(\cdot|\tau(\cdot))}{1-p_{S,k}(\tau(\cdot))}\right]^W$$
$$b(U) = e^{-\lambda_{B,k}}\lambda_{B,k}^{|U|}p_{B,k}^U$$

where $h^X \triangleq \prod_{x\in X} h(x)$ with $h^\emptyset \triangleq 1$ for a real valued function h, and $\mathcal{T}(W, X_{k-1})$ denotes the set of all one-to-one functions taking a finite set W to a finite set X_{k-1} with the convention that the summation over $\mathcal{T}(W, X_{k-1})$ is zero when $|W| > |X_{k-1}|$ and unity when $W = \emptyset$. For a formulation involving labels or identities which permits joint estimation of sources and trajectories see [20]. Generalizations to multi dimensional (non-DOA) states are routine.

4. CONCLUSION

A new framework for array signal processing has been proposed, which is able to accommodate an unknown and time varying number of moving signal and interference sources as well as uncertainties in the array parameters via stochastic geometric modeling, and is amenable to recursive and online execution via Bayesian inferencing.

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